10.6. Particles in the n^{th} Landau Level

2nd Quantized Form

For many particles, the 2nd quantized form is more convenient.

$$H = \int d^2 r\, \mathcal{H}$$

$$\mathcal{H} = \frac{1}{2} M\, \Phi^+(\boldsymbol{r})\, \boldsymbol{v}^2\, \Phi(\boldsymbol{r})$$

$$= \hbar\,\omega_c\, \Phi^+(\boldsymbol{r}) \left(a^+ a + \frac{1}{2} \right) \Phi(\boldsymbol{r}) \qquad\qquad \omega_c = \frac{|\,e\,|\; B}{M\,c}$$

where (see 10.2._CyclotronMotion.pdf)

$$a = \sqrt{\frac{M}{2\,\hbar\,\omega_c}}\; (v_1 + i\,e\,v_2)$$

$$= \frac{l_B\, M}{\sqrt{2}\;\hbar} (v_1 + i\,e\,v_2) \qquad\qquad l_B^2 = \frac{\hbar\,c}{|\,e\,|\;B} = \frac{\hbar}{\omega_c\,M}$$

$$a^+ = \frac{l_B\, M}{\sqrt{2}\;\hbar} (v_1 - i\,e\,v_2)$$

$$\therefore \qquad H = \hbar\,\omega_c \int d^2 r\, \Phi^+(\boldsymbol{r})\, a^+ a\, \Phi(\boldsymbol{r}) + \frac{1}{2} N\,\hbar\,\omega_c$$

where

$$N = \int d^2 r\, \Phi^+(\boldsymbol{r})\, \Phi(\boldsymbol{r}) = \#\text{ of particles}$$

Reminder: In this form, Φ & Φ^+ are operators on the many-body Fock space, while a & a^+ are operators on functions. For example, given a complete set of wave functions ϕ_α, we can write

$$\Phi(\boldsymbol{r}) = \sum_\alpha \phi_\alpha(\boldsymbol{r})\, c_\alpha = \sum_\alpha \langle\, \boldsymbol{r} \mid \alpha \,\rangle\, c_\alpha$$

$$\left[c_\alpha\,,\; c_\beta^+ \right]_\mp = \delta_{\alpha\beta} \qquad \text{for} \quad \begin{matrix}\text{bosons}\\ \text{fermions}\end{matrix}$$

then

$$a\,\Phi(\boldsymbol{r}) \;|\, n_1, n_2, \;\ldots \rangle\rangle = \sum_\alpha \left[a\,\phi_\alpha(\boldsymbol{r}) \right] c_\alpha \left| n_1, n_2, \;\ldots \right\rangle\!\!\right\rangle$$

$$= \sum_\alpha \left[a\,\phi_\alpha(\boldsymbol{r}) \right] \sqrt{n_\alpha}\; (\pm)^{s_\alpha} \left| \;\ldots.,\; n_\alpha - 1,\; \ldots \right\rangle\!\!\right\rangle$$

where $\qquad s_\alpha = \sum_{\beta=1}^{\alpha-1} n_\beta$

& $|\, n_1, n_2, \;\ldots \rangle\rangle$ is the properly symmetrized (bosonic or fermionic) Fock state with n_α particles occupying the 1-particle state $|\,\alpha\,\rangle$.

Alternatively, $\Phi^+(\boldsymbol{r})$ creates a particle at $\boldsymbol{r}$ so that

$$\Phi^+(\boldsymbol{r}_1) \ldots \Phi^+(\boldsymbol{r}_N) \mid 0 \rangle\rangle = \mid \boldsymbol{r}_1, \;\ldots,\; \boldsymbol{r}_N \rangle\rangle$$

$$\rightarrow \qquad \langle\langle\, \boldsymbol{r}_1, \;\ldots,\; \boldsymbol{r}_N \mid\; = \langle\langle\, 0 \mid \Phi(\boldsymbol{r}_N) \ldots \Phi(\boldsymbol{r}_1)$$

The symmetrized wave function for an N-particle system with n_α particles in the α state is given by

$$\Phi(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \langle\langle \mathbf{r}_1, \ldots, \mathbf{r}_N \mid n_1, n_2, \ldots \rangle\rangle \qquad \sum_\alpha n_\alpha = N$$

$$= \langle\langle 0 \mid \Phi(\mathbf{r}_N) \ldots \Phi(\mathbf{r}_1) \mid n_1, n_2, \ldots \rangle\rangle$$

For example, for 2 particles in 2 states a & b, we have

$$\Phi(\mathbf{r}_1, \mathbf{r}_2) = \sum_{\alpha,\beta} \phi_\alpha(\mathbf{r}_2)\, \phi_\beta(\mathbf{r}_1) \langle\langle 0 \mid c_\alpha\, c_\beta \mid n_a = 1, n_b = 1 \rangle\rangle$$

$$= \sum_\alpha \phi_\alpha(\mathbf{r}_2) \{ \phi_a(\mathbf{r}_1) \langle\langle 0 \mid c_\alpha \mid 01 \rangle\rangle \pm \phi_b(\mathbf{r}_1) \langle\langle 0 \mid c_\alpha \mid 10 \rangle\rangle \}$$

$$= \phi_a(\mathbf{r}_1)\, \phi_b(\mathbf{r}_2) \pm \phi_b(\mathbf{r}_1)\, \phi_a(\mathbf{r}_2) \qquad \text{for} \quad \begin{matrix} \text{bosons} \\ \text{fermions} \end{matrix}$$

Symmetric Gauge

Consider now an expansion in terms of the Landau states. In the symmetric gauge, we have (see 10.3._SymmetricGauge.pdf),

$$\phi_\alpha(\mathbf{r}) = \Psi_{nm}(\mathbf{r}) = \langle \mathbf{r} \mid n\, m \rangle$$

$$\Phi(\mathbf{r}) = \sum_{nm} \Psi_{nm}(\mathbf{r})\, c_{nm} = \sum_{nm} \langle \mathbf{r} \mid n\, m \rangle\, c_{nm}$$

with

$$\Psi_{nm}(\mathbf{r}) = \frac{1}{\sqrt{(n+m)!\, 2\pi\, l_B^2}} \left(\frac{i}{\sqrt{2}} \tilde{z}^* \right)^n \left(\frac{\tilde{z}}{\sqrt{2}} \right)^m \exp\left(-\frac{1}{4} \tilde{z}\, \tilde{z}^* \right)$$

$$= \frac{i^n}{\sqrt{(n+m)!\, 2\pi\, l_B^2}} \left(\frac{\tilde{r}}{\sqrt{2}} \right)^{n+m} e^{i\,\mathrm{e}\,(m-n)\,\theta} \exp\left(-\frac{1}{4} \tilde{r}^2 \right)$$

Reminder: Ψ_{nm} describes a particle with cyclotron energy $E_{nm} = \hbar\omega_c\left(n + \frac{1}{2}\right)$ & guiding center on the circle of radius $R = \sqrt{2\, m + 1}\; l_B$ about the origin.

The guiding center obviously cannot be outside the system. Thus, for a disk sample of radius L, we have

$$m_{\max} \simeq \text{Floor}\left\{ \frac{1}{2} \left[\left(\frac{L}{l_B} \right)^2 - 1 \right] \right\} \simeq \text{Floor}\left(\frac{L}{\sqrt{2}\, l_B} \right)^2$$

Since E_{nm} is independent of m, all states $\mid n\, m \rangle$ for a fixed n have the same probability of being occupied. Thus, for N particles in the n^{th} Landau level, there're

$$N_C = \frac{N!}{m_{\max}!\, (N - m_{\max})!}$$

distinct configurations. The corresponding N-particle wave function is then the linear combination of the properly symmetrized wave functions for each configuration. The advantage of the 2nd quantization scheme is that one can avoid dealing with such a mess.

There're irreconcilable differences between Ezawa's & our versions of the many body problem. The following is the best we can do to formulate Ezawa's eqs(10.6.12-29)

$$\Psi_{nm}(\mathbf{r}) = \langle \mathbf{r} \mid n\, m \rangle$$

$$\sum_{nm} \mid n\, m \rangle \langle n\, m \mid = 1$$

$$\Phi(\mathbf{r}) = \sum_{nm} \Psi_{nm}(\mathbf{r})\, c_{nm} = \sum_{nm} \langle \mathbf{r} \mid n\, m \rangle\, c_{nm}$$

Comment: In our version, $| \Phi \rangle$ is meaningless since Φ, by virtue of c_{nm}, is an operator on the states $| \ldots \rangle\rangle$.

$$\int d^2 r \, \Psi^*_{nm}(\boldsymbol{r}) \, \Psi_{n'm'}(\boldsymbol{r}) = \delta_{nn'} \, \delta_{mm'}$$

$$= \int d^2 r \, \langle n m \mid \boldsymbol{r} \rangle \langle \boldsymbol{r} \mid n' m' \rangle$$

→ $$\int d^2 r \, \Psi^*_{nm}(\boldsymbol{r}) \, \Phi(\boldsymbol{r}) = \sum_{n'm'} \int d^2 r \, \Psi^*_{nm}(\boldsymbol{r}) \, \Psi_{n'm'}(\boldsymbol{r}) \, c_{n'm'} = c_{nm}$$

or $$c_{nm} = \int d^2 r \, \langle n m \mid \boldsymbol{r} \rangle \, \Phi(\boldsymbol{r})$$

Going back to the Lagrangian formulation (see 3.2._SchrodingerField.pdf), one gets from canonical quantization that

$$[\, \Phi(t, \boldsymbol{r}) \, , \; \Phi^+(t, \; \boldsymbol{r}') \,]_\mp = \delta(\boldsymbol{r} - \boldsymbol{r}')$$

→ $$[\, c_{nm}, \, c_{n'm'}]_\mp = \delta_{nn'} \, \delta_{mm'}$$

For an N-particle state in the n^{th} Landau level specified by the configuration

$$\{ n_{nm} \} = (n_{n0}, \, n_{n1}, \; \ldots, \, n_{n,m_{\max}}) \qquad \text{with} \qquad \sum_m n_{nm} = N$$

we have

$$| \; \{ n_{nm} \} \; \rangle\rangle = \prod_{m = m_{\max}}^{0} \frac{c^{+\, n_{nm}}_{nm}}{\sqrt{n_{nm}\,!}} \left| \, 0 \right\rangle\!\!\right\rangle$$

$$= \prod_{m = m_{\max}}^{0} \frac{1}{\sqrt{n_{nm}\,!}} \prod_{j=1}^{n_{nm}} \int d^2 r_j \, \Psi^*_{nm}(\boldsymbol{r}_j) \, \Phi^+(\boldsymbol{r}_j) \left| \, 0 \right\rangle\!\!\right\rangle$$

The Hilbert sub-space $\mathbb{H}_n$ of the n^{th} Landau level is spanned by the basis

$$\{ \; | \; n m \rangle ; \; m = 0, \, 1, \, 2, \; \ldots \, \}$$

The n^{th} level projection operator is therefore

$$\mathbb{P}_n = \sum_{m=0}^{m_{\max}} | \; n m \rangle \langle n m \; |$$

Thus,

$$\mathbb{P}_n^2 = \sum_{m,m'} | \; n m \rangle \langle n m \mid n m' \rangle \langle n m' \; | = \sum_m | \; n m \rangle \langle n m \; | = \mathbb{P}_n$$

as befits a projection operator.

$$\Phi(\boldsymbol{r}) = \sum_{nm} \langle \boldsymbol{r} \mid n m \rangle \, c_{nm}$$

→ $$\mathbb{P}_n \, \Phi(\boldsymbol{r}) = \sum_{n'm'} \langle \boldsymbol{r} \mid \mathbb{P}_n \mid n' m' \rangle \, c_{n'm'}$$

$$= \sum_{n'm'm} \langle \boldsymbol{r} \mid n m \rangle \langle n m \mid n' m' \rangle \, c_{n'm'}$$

$$= \sum_m \langle \boldsymbol{r} \mid n m \rangle \, c_{nm} = \sum_m \Psi_{nm}(\boldsymbol{r}) \, c_{nm}$$

$$\equiv \Phi_n(\boldsymbol{r})$$

$$[\, \Phi_0(\boldsymbol{r}), \, \Phi^+_0(\boldsymbol{r}') \,]_\mp = \sum_{mm'} \Psi_{0m}(\boldsymbol{r}) \, \Psi^*_{0m'}(\boldsymbol{r}') \, [\, c_{0m} \, , \; c^+_{0m'}]_\mp$$

$$= \sum_m \Psi_{0m}(\boldsymbol{r}) \, \Psi^*_{0m}(\boldsymbol{r}') = \sum_m \langle \boldsymbol{r} \mid 0 m \rangle \langle 0 m \mid \boldsymbol{r}' \rangle$$

$$\Psi_{0m}(\boldsymbol{r}) = \frac{1}{\sqrt{m!\,2\pi\,l_B^2}}\left(\frac{1}{\sqrt{2}}\tilde{r}\right)^m e^{i\,e\,m\,\theta}\exp\left(-\frac{1}{4}\tilde{r}^2\right)$$

$$\rightarrow \quad \Psi_{0m}(\boldsymbol{r})\,\Psi^*_{0m}(\boldsymbol{r}') = \frac{1}{m!\,2\pi\,l_B^2}\left(\frac{\tilde{r}\,\tilde{r}'}{2}\right)^m e^{i\,e\,m(\theta-\theta')}\exp\left(-\frac{1}{4}(\tilde{r}^2+\tilde{r}'^2)\right)$$

$$\therefore \quad [\Phi_0(\boldsymbol{r}),\,\Phi_0^+(\boldsymbol{r}')]_\mp = \frac{1}{2\pi\,l_B^2}\left(\sum_m \frac{(\tilde{r}\,\tilde{r}')^m e^{i\,e\,m(\theta-\theta')}}{2^m\,m!}\right)\exp\left(-\frac{1}{4}(\tilde{r}^2+\tilde{r}'^2)\right)$$

$$= \frac{1}{2\pi\,l_B^2}\exp\left(\frac{1}{2}\tilde{r}\,\tilde{r}'\,e^{i\,e(\theta-\theta')} - \frac{1}{4}(\tilde{r}^2+\tilde{r}'^2)\right)$$

$$\tilde{r}\,e^{i\,e\,\theta} = \tilde{r}\,(\cos\theta + i\,e\,\sin\theta) = \tilde{x} + i\,e\,\tilde{y}$$

$$\rightarrow \quad \tilde{r}\,\tilde{r}'\,e^{i\,e(\theta-\theta')} - \frac{1}{2}(\tilde{r}^2+\tilde{r}'^2) = (\tilde{x}+i\,e\,\tilde{y})(\tilde{x}'-i\,e\,\tilde{y}') - \frac{1}{2}(\tilde{x}^2+\tilde{y}^2+\tilde{x}'^2+\tilde{y}'^2)$$

$$= \tilde{x}\,\tilde{x}' + \tilde{y}\,\tilde{y}' - i\,e\,(\tilde{x}\,\tilde{y}' - \tilde{y}\,\tilde{x}') - \frac{1}{2}(\tilde{x}^2+\tilde{y}^2+\tilde{x}'^2+\tilde{y}'^2)$$

$$= -\frac{1}{2}\left[(\tilde{x}-\tilde{x}')^2 + (\tilde{y}-\tilde{y}')^2\right] - i\,e\,(\tilde{x}\,\tilde{y}' - \tilde{y}\,\tilde{x}')$$

$$= -\frac{1}{2}(\tilde{\boldsymbol{r}}-\tilde{\boldsymbol{r}}')^2 - i\,e\,(\tilde{\boldsymbol{r}}\times\tilde{\boldsymbol{r}}')_3$$

$$\therefore \quad [\Phi_0(\boldsymbol{r}),\,\Phi_0^+(\boldsymbol{r}')]_\mp = \frac{1}{2\pi\,l_B^2}\exp\left(-\frac{1}{4}(\tilde{\boldsymbol{r}}-\tilde{\boldsymbol{r}}')^2 - \frac{1}{2}i\,e\,(\tilde{\boldsymbol{r}}\times\tilde{\boldsymbol{r}}')_3\right)$$

Comparing with

$$[\Phi_0(\boldsymbol{r}),\,\Phi_0^+(\boldsymbol{r}')]_\mp = \sum_m \langle \boldsymbol{r} \mid 0\,m\rangle\langle 0\,m \mid \boldsymbol{r}'\rangle$$

we see that

$$\sum_m \mid 0\,m\rangle\langle 0\,m \mid \;\neq 1$$

so that the basis $\{\,\mid 0\,m\rangle\}$ is not complete.

Landau Gauge

In the Landau gauge, the eigenstates are given by (see 10.4._LandauGauge.pdf)

$$\Psi_{k\,n}(\boldsymbol{r}) = \langle \boldsymbol{r} \mid k\,n\rangle$$

For the lowest ($n=0$) Landau level with box normalization,

$$\Psi_{k\,0}(\boldsymbol{r}) = \frac{1}{\sqrt{\pi^{1/2}\,l_B}}\frac{e^{i\,k\,x}}{\sqrt{L_x}}\exp\left\{-\frac{1}{2}\left(\frac{y-Y}{l_B}\right)^2\right\} \qquad (\,Y = -e\,l_B^2\,k\,)$$

so that

$$\langle k\,0 \mid k'\,0\rangle = \delta_{k\,k'}$$

Thus,

$$\Phi(\boldsymbol{r}) = \sum_{k\,n}\Psi_{k\,n}(\boldsymbol{r})\,c_{k\,n}$$

$$[c_{k\,n},\; c^+_{k'\,n'}]_\mp = \delta_{k\,k'}\,\delta_{n\,n'}$$

Projection onto the $n=0$ level gives

$$\Phi_0(\boldsymbol{r}) = \sum_k \Psi_{k\,0}(\boldsymbol{r})\,c_{k\,0}$$

$$[c_{k0},\ c^+_{k'0}]_\mp = \delta_{kk'}$$

$$[\Phi_0(\boldsymbol{r}),\ \Phi^+_0(\boldsymbol{r}')]_\mp = \sum_{kk'} \Psi_{k0}(\boldsymbol{r})\, \Psi^*_{k'0}(\boldsymbol{r}')\, [c_{k0},\ c^+_{k'0}]_\mp$$

$$= \sum_k \Psi_{k0}(\boldsymbol{r})\, \Psi^*_{k0}(\boldsymbol{r}') = \sum_k \langle \boldsymbol{r} \mid k\,0 \rangle \langle k\,0 \mid \boldsymbol{r}' \rangle$$

$$\Psi_{k0}(\boldsymbol{r})\, \Psi^*_{k0}(\boldsymbol{r}') = \frac{1}{\pi^{1/2}\, l_B} \frac{e^{i k (x-x')}}{L_x} \exp\left\{ -\frac{1}{2}\left[\left(\frac{y-Y}{l_B}\right)^2 + \left(\frac{y'-Y}{l_B}\right)^2 \right] \right\}$$

$$\sum_k e^{i k (x-x')} \simeq \frac{L_x}{2\pi} \int d k\, e^{i k (x-x')} = L_x\, \delta(x-x')$$

$$\to \qquad [\Phi_0(\boldsymbol{r}),\ \Phi^+_0(\boldsymbol{r}')]_\mp = \frac{1}{\pi^{1/2}\, l_B}\, \delta(x-x') \exp\left\{ -\frac{1}{2}\left[\left(\frac{y-Y}{l_B}\right)^2 + \left(\frac{y'-Y}{l_B}\right)^2 \right] \right\}$$

Coherent States

Consider the projection of $\mid \boldsymbol{r} \rangle$ onto the n^{th} Landau level subspace $\mathbb{H}_n$,

$$\mid \mathbb{r}_n \rangle = \mathbb{P}_n \mid \boldsymbol{r} \rangle = \sum_{m=0}^{m_{\max}} \mid n\,m \rangle \langle n\,m \mid \boldsymbol{r} \rangle$$

$$\to \qquad \langle \mathbb{r}_n \mid \mathbb{r}_n' \rangle = \langle \boldsymbol{r} \mid \mathbb{P}_n \mathbb{P}_n \mid \boldsymbol{r}' \rangle = \langle \boldsymbol{r} \mid \mathbb{P}_n \mid \boldsymbol{r}' \rangle$$

$$= \sum_m \langle \boldsymbol{r} \mid n\,m \rangle \langle n\,m \mid \boldsymbol{r}' \rangle$$

For $n = 0$,

$$\langle \mathbb{r}_0 \mid \mathbb{r}_0' \rangle = \sum_m \langle \boldsymbol{r} \mid 0\,m \rangle \langle 0\,m \mid \boldsymbol{r}' \rangle$$

$$= \frac{1}{2\pi\, l_B^2} \exp\left(-\frac{1}{4} (\tilde{\boldsymbol{r}} - \tilde{\boldsymbol{r}}')^2 - \frac{1}{2} i\, \mathbb{e}\, (\tilde{\boldsymbol{r}} \times \tilde{\boldsymbol{r}}')_3 \right)$$

$$\mid \mathbb{r}_0 \rangle \langle \mathbb{r}_0 \mid = \sum_{m m'} \mid 0\,m \rangle \langle 0\,m \mid \boldsymbol{r} \rangle \langle \boldsymbol{r} \mid 0\,m' \rangle \langle 0\,m' \mid$$

$$\int d^2 r \mid \boldsymbol{r} \rangle \langle \boldsymbol{r} \mid = 1$$

$$\to \qquad \int d^2 r \mid \mathbb{r}_0 \rangle \langle \mathbb{r}_0 \mid = \sum_{m m'} \mid 0\,m \rangle \langle 0\,m \mid 0\,m' \rangle \langle 0\,m' \mid = \sum_m \mid 0\,m \rangle \langle 0\,m \mid$$

$$= 1_0 = \text{identity operator in } \mathbb{H}_0$$

i.e., $\{ \mid \mathbb{r}_0 \rangle \}$ is a non-orthogonal, but complete basis for $\mathbb{H}_0$.

In other words,

$$\langle 0\,m \mid \mathbb{r}_0 \rangle = \sum_{m'} \langle 0\,m \mid 0\,m' \rangle \langle 0\,m' \mid \boldsymbol{r} \rangle = \langle 0\,m \mid \boldsymbol{r} \rangle$$

$$\Psi_{0m}(\boldsymbol{r}) = \langle \boldsymbol{r} \mid 0\,m \rangle = \frac{1}{\sqrt{m!\, 2\pi\, l_B^2}} \left(\frac{1}{\sqrt{2}} \tilde{r} \right)^m e^{i\, \mathbb{e}\, m\, \theta} \exp\left(-\frac{1}{4} \tilde{r}^2 \right)$$

$$\to \qquad \mid \mathbb{r}_0 \rangle = \sum_m \mid 0\,m \rangle \langle 0\,m \mid \boldsymbol{r} \rangle$$

$$= \sum_m \mid 0\,m \rangle \frac{1}{\sqrt{m!\, 2\pi\, l_B^2}} \left(\frac{1}{\sqrt{2}} \tilde{\mathbb{z}}^* \right)^m \exp\left(-\frac{1}{4} \tilde{\mathbb{z}}\, \tilde{\mathbb{z}}^* \right)$$

To evaluate the summation, we compare it with (see 1.5._CoherentStatesAndVonNeumannLattice.pdf)

$$| v \rangle = e^{-|v|^2/2} e^{v a^+} \left| 0 \right\rangle = e^{-|v|^2/2} \sum_{n=0}^{\infty} \frac{v^n}{\sqrt{n!}} \left| n \right\rangle$$

Setting

$$v \to \tilde{z}^* \qquad a \to b$$

where b satisfies

$$\frac{b^{+m}}{\sqrt{m!}} | 00 \rangle = | 0m \rangle \qquad \& \qquad [b, b^+] = 1$$

gives (see §10.3)

$$b^+ = \frac{1}{2\sqrt{2}} \left(\tilde{z} - 4 \frac{\partial}{\partial \tilde{z}^*} \right)$$

$$\& \qquad | \mathbb{r}_0 \rangle = \frac{1}{\sqrt{2\pi l_B^2}} \exp\left(-\frac{1}{4} \tilde{r}^2 \right) \sum_{m=0}^{m_{\max}} \frac{\tilde{z}^{*m}}{\sqrt{2^m m!}} \left| 0m \right\rangle$$

$$b \,| 0m \rangle = \sqrt{m} \left| 0, m-1 \right\rangle$$

$$\to \qquad b \,| \mathbb{r}_0 \rangle = \frac{1}{\sqrt{2\pi l_B^2}} \exp\left(-\frac{1}{4} \tilde{r}^2 \right) \sum_{m=1}^{m_{\max}} \frac{\tilde{z}^{*m}}{\sqrt{2^m (m-1)!}} \left| 0, m-1 \right\rangle$$

$$= \frac{1}{\sqrt{2\pi l_B^2}} \exp\left(-\frac{1}{4} \tilde{r}^2 \right) \frac{\tilde{z}^*}{\sqrt{2}} \sum_{m=0}^{m_{\max}} \frac{\tilde{z}^{*m}}{\sqrt{m!}} \left| 0, m \right\rangle$$

$$= \frac{\tilde{z}^*}{\sqrt{2}} \left| \mathbb{r}_0 \right\rangle$$

$\therefore$ $| \mathbb{r}_0 \rangle$ is a coherent state of b with eigenvalue $\dfrac{\tilde{z}^*}{\sqrt{2}}$.

Alternatively, writing

$$| \mathbb{r}_0 \rangle = \frac{1}{\sqrt{2\pi l_B^2}} \exp\left(-\frac{1}{4} \tilde{r}^2 \right) e^{\tilde{z}^* b^+ / \sqrt{2}} \left| 00 \right\rangle$$

& comparing it with the v eq. also gives the same conclusion.

In the $\boldsymbol{r}$- (or z-) representation in which

$$b = \frac{1}{2\sqrt{2}} \left(\tilde{z}^* + 4 \frac{\partial}{\partial \tilde{z}} \right) \qquad b^+ = \frac{1}{2\sqrt{2}} \left(\tilde{z} - 4 \frac{\partial}{\partial \tilde{z}^*} \right)$$

we need to refine our notations & write

$$| \mathbb{r}_0(\boldsymbol{r}) \rangle = \sum_m | 0m \rangle \langle 0m | \boldsymbol{r} \rangle$$

$$= \sum_m | 0m \rangle \frac{1}{\sqrt{m!\, 2\pi l_B^2}} \left(\frac{\tilde{z}^*}{\sqrt{2}} \right)^m \exp\left(-\frac{1}{4} \tilde{z} \tilde{z}^* \right)$$

so that

$$\mathbb{r}_0(\boldsymbol{r}\,;\, \boldsymbol{r}') = \langle \boldsymbol{r} \,|\, \mathbb{r}_0(\boldsymbol{r}') \rangle$$

$$= \sum_m \langle \boldsymbol{r} \,|\, 0m \rangle \langle 0m \,|\, \boldsymbol{r}' \rangle$$

$$= \sum_m \langle \boldsymbol{r} \mid 0\, m \rangle \frac{1}{\sqrt{m!\, 2\pi\, l_B^2}} \left(\frac{\tilde{z}'^*}{\sqrt{2}} \right)^m \exp\left(-\frac{1}{4} \tilde{z}' \tilde{z}'^* \right)$$

$$= \sum_m \frac{1}{m!\, 2\pi\, l_B^2} \left(\frac{\tilde{z}\, \tilde{z}'^*}{2} \right)^m \exp\left(-\frac{1}{4} (\tilde{z}\, \tilde{z}^* + \tilde{z}' \tilde{z}'^*) \right)$$

$$\therefore \quad \langle \boldsymbol{r} \mid b \mid \mathbb{r}_0(\boldsymbol{r}') \rangle = b(\mathbb{z})\, \mathbb{r}_0(\boldsymbol{r}\,;\, \boldsymbol{r}')$$

$$= \frac{1}{2\sqrt{2}} \left(\tilde{z}^* + 4 \frac{\partial}{\partial \tilde{z}} \right) \mathbb{r}_0(\boldsymbol{r}\,;\, \boldsymbol{r}')$$

$$4 \frac{\partial}{\partial \tilde{z}} \left[\left(\frac{\tilde{z}}{\sqrt{2}} \right)^m \exp\left(-\frac{1}{4} \tilde{z}\, \tilde{z}^* \right) \right] = \left(\frac{4m}{\tilde{z}} - \tilde{z}^* \right) \left(\frac{\tilde{z}}{\sqrt{2}} \right)^m \exp\left(-\frac{1}{4} \tilde{z}\, \tilde{z}^* \right)$$

$$\rightarrow \quad \frac{1}{2\sqrt{2}} \left(\tilde{z}^* + 4 \frac{\partial}{\partial \tilde{z}} \right) \left[\left(\frac{\tilde{z}}{\sqrt{2}} \right)^m \exp\left(-\frac{1}{4} \tilde{z}\, \tilde{z}^* \right) \right] = m \left(\frac{\tilde{z}}{\sqrt{2}} \right)^{m-1} \exp\left(-\frac{1}{4} \tilde{z}\, \tilde{z}^* \right)$$

$$\therefore \quad \langle \boldsymbol{r} \mid b \mid \mathbb{r}_0(\boldsymbol{r}') \rangle = \sum_m \frac{1}{m!\, 2\pi\, l_B^2} \left(\frac{\tilde{z}'^*}{\sqrt{2}} \right)^m \exp\left(-\frac{1}{4} \tilde{z}' \tilde{z}'^* \right) m \left(\frac{\tilde{z}}{\sqrt{2}} \right)^{m-1} \exp\left(-\frac{1}{4} \tilde{z}\, \tilde{z}^* \right)$$

$$= \sum_m \frac{\tilde{z}'^*}{(m-1)!\, 2\pi\, l_B^2 \sqrt{2}} \left(\frac{\tilde{z}\, \tilde{z}'^*}{2} \right)^{m-1} \exp\left(-\frac{1}{4} (\tilde{z}\, \tilde{z}^* + \tilde{z}' \tilde{z}'^*) \right)$$

$$= \frac{\tilde{z}'^*}{\sqrt{2}} \mathbb{r}_0(\boldsymbol{r}\,;\, \boldsymbol{r}')$$

i.e., $\mathbb{r}_0(\boldsymbol{r}\,;\, \boldsymbol{r}')$ is a coherent state with eigenvalue $\dfrac{\tilde{z}'^*}{\sqrt{2}}$.