10.7. Hall Current

In the additional presence of a constant in-plane electric field $\boldsymbol{E} = (E_x, E_y, 0)$, the Hamiltonian becomes

$$H = \frac{1}{2} M \boldsymbol{v}^2 - e\, \boldsymbol{r} \cdot \boldsymbol{E}$$

From 10.2._CyclotronMotion.pdf,

$$[\boldsymbol{v}, \boldsymbol{R}] = 0 \qquad\qquad [r_i, R_j] = -i \frac{\hbar c}{e B} \varepsilon_{i j 3} = -i\, e\, l_B^2\, \varepsilon_{i j 3}$$

→ $$\dot{R}_i = \frac{1}{i \hbar} [R_i, H] = -\frac{e E_j}{i \hbar} [R_i, r_j] = -\frac{c}{B} \varepsilon_{i j 3} E_j$$

or $$\dot{\boldsymbol{R}} = \frac{c}{B} \hat{\boldsymbol{z}} \times \boldsymbol{E} = \frac{c}{B} (-E_y, E_x)$$

Equating $\dot{\boldsymbol{R}}$ with the drift velocity $\boldsymbol{v}_D$, the current density is

$$\mathcal{J} = e \rho \boldsymbol{v}_D \qquad\qquad \rho = \psi^* \psi$$

$$= \frac{e \rho c}{B} (-E_y, E_x)$$

In general, the Coulomb potential due to an external electrostatic potential $\frac{1}{c} A_0$ is

$$\frac{e}{c} \int d^2 r\, \rho(\boldsymbol{r})\, A_0(\boldsymbol{r}) \qquad\qquad \left(\boldsymbol{E} = -\frac{1}{c} \nabla A_0 \right)$$

The 2nd quantized Hamiltonian is therefore

$$H = \int d^2 r\, \Phi^+(\boldsymbol{r}) \left(\frac{1}{2} M \boldsymbol{v}^2 + \frac{e}{c} A_0 \right) \Phi(\boldsymbol{r})$$

Current density

$$\boldsymbol{p} = \frac{\hbar}{i} \nabla \quad \to \quad \boldsymbol{p}^+ = -\frac{\hbar}{i} \nabla^+ = -\frac{\hbar}{i} \overleftarrow{\nabla} = \frac{\hbar}{i} \nabla = \boldsymbol{p} \qquad \left(\nabla^+ = \overleftarrow{\nabla} = -\nabla \right)$$

$$\therefore \quad \boldsymbol{D} = \nabla - i \frac{e}{\hbar c} \boldsymbol{A} \quad \to \quad \boldsymbol{D}^+ = \overleftarrow{\nabla} + i \frac{e}{\hbar c} \boldsymbol{A} = \overleftarrow{\boldsymbol{D}^*} = -\nabla + i \frac{e}{\hbar c} \boldsymbol{A} = -\boldsymbol{D}$$

$$\boldsymbol{v} = \frac{\hbar}{M i} \boldsymbol{D} \quad \to \quad \boldsymbol{v}^+ = \boldsymbol{v}$$

$$(\boldsymbol{D} \psi)^+ = \psi^+ \overleftarrow{\boldsymbol{D}^*} = \boldsymbol{D}^* \psi^+$$

From the classical definition of the current density

$$\boldsymbol{J} = e \rho \boldsymbol{v}$$

we may write the quantum version as

$$\boldsymbol{J} = \frac{1}{2} e [\psi^+ \boldsymbol{v} \psi + (\boldsymbol{v} \psi)^+ \psi]$$

$$= \frac{e \hbar}{2 M i} [\psi^+ \boldsymbol{D} \psi - (\boldsymbol{D} \psi)^+ \psi]$$

$$= \frac{e \hbar}{2 M i} [\psi^+ \boldsymbol{D} \psi - (\boldsymbol{D}^* \psi^+) \psi]$$

$$= \boldsymbol{J}^+$$

Note that $\boldsymbol{J}$ equals the Noether current density (see 5.3._InteractionWithMatterField.pdf).

$$\boldsymbol{J} = \frac{e\hbar}{2Mi}\left[\psi^+ \boldsymbol{D}\psi - \left(\boldsymbol{D}^* \psi^+\right)\psi\right]$$
$$= \frac{e\hbar}{2Mi}\left[\psi^+\left(\nabla\psi - i\frac{e}{\hbar c}\boldsymbol{A}\psi\right) - \left(\nabla\psi^+ + i\frac{e}{\hbar c}\boldsymbol{A}\psi^+\right)\psi\right]$$
$$= \frac{e\hbar}{2Mi}\left[\psi^+\nabla\psi - (\nabla\psi^+)\psi - 2i\frac{e}{\hbar c}\boldsymbol{A}\psi^+\psi\right]$$
$$= \frac{e\hbar}{2Mi}\left[\psi^+\nabla\psi - \left(\nabla\psi^+\right)\psi\right] - \frac{e^2}{Mc}\boldsymbol{A}\psi^+\psi$$

Using

$$\nabla\left(\psi^+\psi\right) = \left(\nabla\psi^+\right)\psi + \psi^+\nabla\psi$$

one can also re-write the 2nd line in the $\boldsymbol{J}$ expressions as

$$\boldsymbol{J} = \frac{e\hbar}{2Mi}\left[2\psi^+\left(\nabla\psi - i\frac{e}{\hbar c}\boldsymbol{A}\psi\right) - \nabla\left(\psi^+\psi\right)\right]$$
$$= e\psi^+\boldsymbol{v}\psi + i\frac{e\hbar}{2M}\nabla\rho \qquad (\rho = \psi^+\psi)$$
$$= \boldsymbol{J}^+$$

From 10.3._SymmetricGauge.pdf, we have

$$a = \sqrt{\frac{M}{2\hbar\omega_c}}\left(v_1 + i\,e\,v_2\right) \qquad \& \qquad a\,\psi_{00}(\boldsymbol{r}) = 0$$

$$\rightarrow \qquad \mathbb{J}_+ = J_x + i\,e\,J_y$$
$$= e\psi^+\left(v_1 + i\,e\,v_2\right)\psi + i\frac{e\hbar}{2M}\left(\partial_x + i\,e\,\partial_y\right)\rho$$
$$= e\sqrt{\frac{2\hbar\omega_c}{M}}\,\psi^+ a\,\psi + i\frac{e\hbar}{2M}\left(\partial_x + i\,e\,\partial_y\right)\rho$$

For $\psi = \psi_{00}$,

$$\mathbb{J}_+ = i\frac{e\hbar}{2M}\left(\partial_x + i\,e\,\partial_y\right)\rho_{00}$$
$$\rho_{00}(\boldsymbol{r}) = \frac{1}{2\pi l_B^2}\exp\left(-\frac{r^2}{2l_B^2}\right) \qquad \text{(see §10.3)}$$
$$\partial_x r^2 = 2x \qquad \partial_y r^2 = 2y$$
$$\rightarrow \qquad \mathbb{J}_+ = \frac{e\hbar}{M l_B^2}\left(-i\,x + e\,y\right)\rho_{00}$$
$$\therefore \qquad \boldsymbol{J} = \frac{|\,e\,|\,\hbar}{M l_B^2}\rho_{00}\,(y, -x)$$
$$= \omega_c\,\rho_{00}\,(y, -x)$$
$$= \omega_c\,\rho_{00}\,\boldsymbol{r}\times\hat{\boldsymbol{z}}$$
$$= -\omega_c\,\rho_{00}\,r\,\hat{\boldsymbol{\theta}}$$

which is the same as that given in 10.3._SymmetricGauge.pdf.

Landau Gauge

See 10.4._LandauGauge.pdf.

Let

$$\boldsymbol{A} = B(-y,\, 0,\, 0) \qquad \text{with} \qquad \boldsymbol{E} = (0,\, E,\, 0)$$

$$\rightarrow \quad H = -\frac{\hbar^2}{2M}\left(\partial_x^2 + \partial_y^2 - \frac{1}{l_B^4}\, y^2 + 2\, i\, \frac{e}{l_B^2}\, y\, \partial_x + \frac{2\, e\, E}{\hbar\, \omega_c\, l_B^2}\, y\right) \qquad \omega_c = \frac{|\, e\, |\, B}{M\, c} = \frac{\hbar}{M\, l_B^2}$$

$$\Psi(x,\, y) = e^{i\, k\, x}\, \psi(y)$$

$$H\, \Psi = \mathcal{E}\, \Psi$$

$$\rightarrow \quad -\frac{\hbar^2}{2M}\left[-k^2 + \partial_y^2 - \frac{1}{l_B^4}\, y^2 - 2\, \frac{e}{l_B^2}\left(k - \frac{|\, e\, |\, E}{\hbar\, \omega_c}\right) y\right] \psi = \mathcal{E}\, \psi$$

$$\left\{-\partial_y^2 + \frac{1}{l_B^4}\left[y + e\, l_B^2\left(k - \frac{|\, e\, |\, E}{\hbar\, \omega_c}\right)\right]^2\right\} \psi = \frac{2M}{\hbar^2}\left[\mathcal{E} + \frac{(e\, E\, l_B)^2}{2\, \hbar\, \omega_c}\right] \psi$$

$$\boldsymbol{R} = \left(x - i\, e\, l_B^2\, \partial_y\, ,\; e\, l_B^2\left(-k + \frac{|\, e\, |\, E}{\hbar\, \omega_c}\right)\right) \equiv (X,\, Y)$$

$$\rightarrow \quad \left\{-\partial_y^2 + \frac{1}{l_B^4}\,(y - Y)^2\right\} \psi = \frac{2M}{\hbar^2}\left[\mathcal{E} + \frac{(e\, E\, l_B)^2}{2\, \hbar\, \omega_c}\right] \psi$$

In dimensionless form

$$\left\{-(l_B\, \partial_y)^2 + \left(\frac{y - Y}{l_B}\right)^2\right\} \psi = \frac{2}{\hbar\, \omega_c}\left[\mathcal{E} + \frac{(e\, E\, l_B)^2}{2\, \hbar\, \omega_c}\right] \psi$$

$$\mathcal{E} = \hbar\, \omega_c\left(n + \frac{1}{2}\right) - \frac{(e\, E\, l_B)^2}{2\, \hbar\, \omega_c} \qquad n = 0,\, 1,\, 2,\, \ldots$$

$$= \hbar\, \omega_c\left\{n + \frac{1}{2}\left[1 - \left(\frac{e\, E\, l_B}{\hbar\, \omega_c}\right)^2\right]\right\}$$

The Noether current density is

$$\boldsymbol{J} = i\, \frac{e\, \hbar}{2M}\left[(\boldsymbol{D}\psi)^+\, \psi - \psi^+\, \boldsymbol{D}\psi\right]$$

$$D_x\, \Psi_{k\, n}(\boldsymbol{r}) = \left(\partial_x + i\, \frac{e}{l_B^2}\, y\right) e^{i\, k\, x}\, \psi_n(y)$$

$$= i\left(k + \frac{e}{l_B^2}\, y\right) \Psi_{k\, n}(\boldsymbol{r})$$

$$D_y\, \Psi_{k\, n}(\boldsymbol{r}) = e^{i\, k\, x}\, \partial_y\, \psi_n(y)$$

For $n = 0$ with box normalization

$$\Psi_{k\, 0}(\boldsymbol{r}) = \frac{1}{\sqrt{\pi^{1/2}\, l_B}}\, \frac{e^{i\, k\, x}}{\sqrt{L_x}}\, \exp\left\{-\frac{1}{2}\left(\frac{y - Y}{l_B}\right)^2\right\} \qquad Y = e\, l_B^2\left(-k + \frac{|\, e\, |\, E}{\hbar\, \omega_c}\right)$$

$$J_x = \frac{e\, \hbar}{M}\left(k + \frac{e}{l_B^2}\, y\right) \Psi_{k\, 0}^*\, \Psi_{k\, 0}$$

$$= \frac{e\, \hbar}{M}\left(\frac{|\, e\, |\, E}{\hbar\, \omega_c} + e\, \frac{y - Y}{l_B^2}\right) \Psi_{k\, 0}^*\, \Psi_{k\, 0}$$

$$= \frac{e\hbar}{M\, l_B \sqrt{\pi}\, L_x} \left(\frac{|e|E}{\hbar\,\omega_c} + e\,\frac{y-Y}{l_B^2} \right) \exp\left\{ -\left(\frac{y-Y}{l_B} \right)^2 \right\}$$

$$\rightarrow \qquad \langle J_x \rangle = \int_0^{L_x} dx \int_{-\infty}^{\infty} dy\, J_x$$

$$= \frac{e\hbar}{M\, l_B \sqrt{\pi}} \frac{|e|E}{\hbar\,\omega_c} \int_{-\infty}^{\infty} dy \exp\left\{ -\left(\frac{y-Y}{l_B} \right)^2 \right\}$$

$$= \frac{e\,|e|\,E}{M\,\omega_c}$$

$$= e\,\frac{E}{B}\,c$$

which agrees with the classical results.

$$J_y = i\,\frac{e\hbar}{2M} \left\{ -\left(\frac{y-Y}{l_B} \right) + \left(\frac{y-Y}{l_B} \right) \right\} \Psi_{k0}^* \Psi_{k0} = 0$$

Field Operators

For $n = 0$ (see 10.6._ParticlesInTheNthLandauLevel.pdf),

$$\Psi_{k0}(\mathbf{r}) = \frac{1}{\sqrt{\pi^{1/2}\, l_B}} \frac{e^{ikx}}{\sqrt{L_x}} \exp\left\{ -\frac{1}{2} \left(\frac{y-Y}{l_B} \right)^2 \right\} \qquad \left(Y = e\, l_B^2 \left(-k + \frac{|e|E}{\hbar\,\omega_c} \right) \right)$$

$$\rightarrow \qquad \Phi_0(\mathbf{r}) = \sum_k \Phi_{k0}(\mathbf{r}) = \sum_k \Psi_{k0}(\mathbf{r})\, c_{k0}$$

$$= \frac{1}{\sqrt{\pi^{1/2}\, l_B\, L_x}} \sum_k e^{ikx} \exp\left\{ -\frac{1}{2} \left(\frac{y-Y}{l_B} \right)^2 \right\} c_{k0}$$

$$\rho_{k0} = \Psi_{k0}^* \Psi_{k0} = \frac{1}{\pi^{1/2}\, l_B\, L_x} \exp\left\{ -\left(\frac{y-Y}{l_B} \right)^2 \right\}$$

$$\rightarrow \qquad \partial_x \rho_{k0} = 0 \qquad\qquad \partial_y \rho_{k0} = -2\,\frac{y-Y}{l_B^2}\,\rho_{k0}$$

$$\partial_x \Psi_{k0} = i\,k\,\Psi_{k0} \qquad\qquad \partial_y \Psi_{k0} = -\frac{y-Y}{l_B^2}\,\Psi_{k0}$$

$$\therefore \qquad J_x = \frac{e\hbar}{M} \left(\frac{|e|E}{\hbar\,\omega_c} + e\,\frac{y-Y}{l_B^2} \right) \Psi_{k0}^* \Psi_{k0}$$

$$= \frac{e\hbar}{M} \left(\frac{|e|E}{\hbar\,\omega_c} - \frac{1}{2}\, e\, \partial_y \right) \rho_{k0}$$

$$= \frac{e\hbar}{M} \Psi_{k0}^* \left(\frac{|e|E}{\hbar\,\omega_c} - e\, \partial_y \right) \Psi_{k0}$$

In 2nd quantized form

$$J_x = \frac{e\hbar}{M} \left(\frac{|e|E}{\hbar\,\omega_c} - \frac{1}{2}\, e\, \partial_y \right) \Phi_{k0}^+ \Phi_{k0}$$

$$= \frac{e\hbar}{M} \Phi_{k0}^+ \left(\frac{|e|E}{\hbar\,\omega_c} - e\, \partial_y \right) \Phi_{k0}$$

Note that by definition

$$\boldsymbol{J} = \frac{e\hbar}{2Mi}[\Phi^{+}\boldsymbol{D}\Phi - (\boldsymbol{D}\Phi)^{+}\Phi]$$