11.1. Incompressibility

Consider a 2-D system specified thermodynamically by

$$dU = -P\,d\sigma + \mu\,dN + \ldots$$

where U is the energy, P the pressure, σ the area, μ the chemical potential, & N the number of particles.

The compressibility is defined as

$$\kappa_N = -\frac{1}{\sigma}\left(\frac{\partial \sigma}{\partial P}\right)_N$$

Incompressibility means $\kappa = 0$.

$$P = -\left(\frac{\partial U}{\partial \sigma}\right)_N$$

$$\rightarrow \qquad \kappa_N^{-1} = -\sigma\left(\frac{\partial P}{\partial \sigma}\right)_N = \sigma\left(\frac{\partial^2 U}{\partial \sigma^2}\right)_N$$

For non-interacting particles,

$$U = N\,\varepsilon(\rho)$$

where

$\varepsilon =$ energy per particle

$\rho = \dfrac{N}{\sigma} =$ number density

$$\rightarrow \qquad \kappa_N^{-1} = \frac{N}{\rho}\left(\frac{\partial^2 (N\,\varepsilon)}{\partial \left(\frac{N}{\rho}\right)^2}\right)_N = \frac{1}{\rho}\left(\frac{\partial^2 \varepsilon}{\partial (\rho^{-1})^2}\right)_N$$

$$\frac{\partial}{\partial (\rho^{-1})} = -\rho^2 \frac{\partial}{\partial \rho}$$

$$\rightarrow \qquad \frac{\partial^2}{\partial (\rho^{-1})^2} = \rho^2 \frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial}{\partial \rho}\right) = \rho^2\left(2\rho \frac{\partial}{\partial \rho} + \rho^2 \frac{\partial^2}{\partial \rho^2}\right)$$

$$= \rho^3 \frac{\partial}{\partial \rho}\left(1 + \rho \frac{\partial}{\partial \rho}\right) = \rho^2 \frac{\partial^2}{\partial \rho^2}\rho$$

$$\therefore \qquad \kappa_N^{-1} = \rho^2\left(2\frac{\partial \varepsilon}{\partial \rho} + \rho \frac{\partial^2 \varepsilon}{\partial \rho^2}\right)_N$$

$$= \rho^2\left(\frac{\partial^2 (\rho\,\varepsilon)}{\partial \rho^2}\right)_N$$

$$\mu = \left(\frac{\partial U}{\partial N}\right)_\sigma = \left(\frac{\partial (N\,\varepsilon)}{\partial N}\right)_\sigma = \left(\frac{\partial (\rho\,\sigma\,\varepsilon)}{\sigma\,\partial \rho}\right)_\sigma = \left(\frac{\partial (\rho\,\varepsilon)}{\partial \rho}\right)_\sigma$$

$$\rho = \frac{N}{\sigma} \quad \& \quad \varepsilon = \varepsilon(\rho)$$

$\rightarrow \qquad \dfrac{\partial (\rho\,\varepsilon)}{\partial \rho}$ depends on ρ only

$$\therefore \qquad \left(\frac{\partial (\rho\,\varepsilon)}{\partial \rho}\right)_\sigma = \left(\frac{\partial (\rho\,\varepsilon)}{\partial \rho}\right)_N$$

$$\rightarrow \qquad \kappa_N^{-1} = \rho^2 \left(\frac{\partial}{\partial \rho} \left(\frac{\partial (\rho\, \varepsilon)}{\partial \rho} \right)_N \right)_N$$

$$= \rho^2 \left(\frac{\partial}{\partial \rho} \left(\frac{\partial (\rho\, \varepsilon)}{\partial \rho} \right)_\sigma \right)_N = \rho^2 \left(\frac{\partial \mu}{\partial \rho} \right)_{N \text{ or } \sigma}$$

Incompressibilty $\rightarrow$ $\left(\frac{\partial \mu}{\partial \rho} \right)_{N \text{ or } \sigma} = \infty$

which happens if μ is not continuous.