## 11.2. Integer Quantum Hall Effects

Spin-frozen system: All spins are frozen by the Zeeman effect (no spin dynamics).

Filling factor $\nu = 1$:

→ For fermions, the lowest Landau level ($n = 0$) is filled.

i.e. $$N = m_{\max} + 1 \simeq \text{Floor}\left( \frac{L}{\sqrt{2}\, l_B} \right)^2$$

where $L$ is the smaller linear dimension of a rectangular system (see 10.6._ParticlesInTheNthLandauLevel.pdf ).

The wave function is the Slater determinant

$$\Psi(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \Psi_{00}(\boldsymbol{r}_1) & \Psi_{00}(\boldsymbol{r}_2) & \ldots & \Psi_{00}(\boldsymbol{r}_N) \\ \Psi_{01}(\boldsymbol{r}_1) & \Psi_{01}(\boldsymbol{r}_2) & \ldots & \Psi_{01}(\boldsymbol{r}_N) \\ \vdots & \vdots & \ddots & \vdots \\ \Psi_{0\,m_{\max}}(\boldsymbol{r}_1) & \Psi_{0\,m_{\max}}(\boldsymbol{r}_2) & \ldots & \Psi_{0\,m_{\max}}(\boldsymbol{r}_N) \end{vmatrix}$$

Using (see 10.3._SymmetricGauge.pdf )

$$\Psi_{0\,m}(\mathbb{z}) = \frac{1}{\sqrt{m!\, 2\pi\, l_B^2}} \left( \frac{\mathbb{z}}{\sqrt{2}\, l_B} \right)^m \exp\left( -\frac{1}{4\, l_B^2}\, \mathbb{z}\, \mathbb{z}^* \right)$$

→ $$\Psi(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) = C_N \begin{vmatrix} 1 & 1 & \ldots & 1 \\ \mathbb{z}_1 & \mathbb{z}_2 & \ldots & \mathbb{z}_N \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{z}_1^{m_{\max}} & \mathbb{z}_2^{m_{\max}} & \ldots & \mathbb{z}_N^{m_{\max}} \end{vmatrix} \exp\left( -\frac{1}{4\, l_B^2} \sum_{j=1}^{N} \mathbb{z}_j\, \mathbb{z}_j^* \right)$$

where

$$C_N^{-2} = N!\, (2\pi\, l_B^2)^N \prod_{m=0}^{m_{\max}} m!\, (2\, l_B^2)^m \qquad\qquad m_{\max} = N - 1$$

In general

$$A(n) = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ a_1 & a_2 & \ldots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{n-1} & a_2^{n-1} & \ldots & a_n^{n-1} \end{vmatrix}$$

$= \text{signed sum of monomials } a_1^{m_1}\, a_2^{m_2} \ldots a_n^{m_n}$

with $$\sum_{j=1}^{n-1} m_j = \sum_{k=1}^{n-1} k = \frac{1}{2} n (n-1) \qquad \& \qquad 0 \le m_j \le n-1$$

Also, interchanging any pair of variables, e.g., $a_j \leftrightarrow a_k$ changes the sign of $A(n)$.

It's easy to check that these conditions are met by

$$A(n) = \prod_{j>k}^{n} (a_j - a_k) = \prod_{j=2}^{n} \prod_{k=1}^{j-1} (a_j - a_k)$$

For example, there are $\frac{1}{2} n(n-1)$ factors in the product so that the condition on $\sum_{j=1}^{n-1} m_j$ is met.

Since each $(j, k)$ pair of indices appears in the product exactly once, $0 \le m_j \le n-1$ is satisfied.

Finally, the sign change of $A$ is trivially satisfied.

$$\therefore \qquad \Psi(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) = C_N \left( \prod_{j>k} (z_j - z_k) \right) \exp\left( -\frac{1}{4 l_B^2} \sum_{j=1}^{N} z_j z_j^* \right)$$

A hole at the central ($m = 0$) ring is an $N - 1$ particle system with the state $|\,00\rangle$ unoccupied. Its wave function is therefore

$$\Psi^{(0)}(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_{N-1}) = \frac{1}{\sqrt{(N-1)!}} \begin{vmatrix} \Psi_{01}(\boldsymbol{r}_1) & \Psi_{01}(\boldsymbol{r}_2) & \ldots & \Psi_{01}(\boldsymbol{r}_{N-1}) \\ \Psi_{02}(\boldsymbol{r}_1) & \Psi_{02}(\boldsymbol{r}_2) & \ldots & \Psi_{02}(\boldsymbol{r}_{N-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \Psi_{0\,m_{\max}}(\boldsymbol{r}_1) & \Psi_{0\,m_{\max}}(\boldsymbol{r}_2) & \ldots & \Psi_{0\,m_{\max}}(\boldsymbol{r}_{N-1}) \end{vmatrix}$$

$$= C_{N-1} \begin{vmatrix} z_1 & z_2 & \ldots & z_{N-1} \\ z_1^2 & z_2^2 & \ldots & z_{N-1}^2 \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{m_{\max}} & z_2^{m_{\max}} & \ldots & z_{N-1}^{m_{\max}} \end{vmatrix} \exp\left( -\frac{1}{4 l_B^2} \sum_{j=1}^{N-1} z_j z_j^* \right)$$

Let

$$A^{(0)}(n) = \begin{vmatrix} a_1 & a_2 & \ldots & a_n \\ a_1^2 & a_2^2 & \ldots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^n & a_2^n & \ldots & a_n^n \end{vmatrix}$$

From the Laplace expansion of a determinant $A$, we see that multiplying a row, or a column, of $A$ by $c$ changes the value of $A$ to $c\,A$. Thus

$$A^{(0)}(n) = a_1 \begin{vmatrix} 1 & a_2 & \ldots & a_n \\ a_1 & a_2^2 & \ldots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{n-1} & a_2^n & \ldots & a_n^n \end{vmatrix} = a_1 a_2 \begin{vmatrix} 1 & 1 & \ldots & a_n \\ a_1 & a_2 & \ldots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{n-1} & a_2^{n-1} & \ldots & a_n^n \end{vmatrix}$$

$$= \prod_{k=1}^{n} a_k\, A(n)$$

$$\therefore \qquad \Psi^{(0)}(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_{N-1}) = C_{N-1} \left( \prod_{k=1}^{N-1} z_k \right) \begin{vmatrix} 1 & 1 & \ldots & 1 \\ z_1 & z_2 & \ldots & z_{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{m_{\max}-1} & z_2^{m_{\max}-1} & \ldots & z_{N-1}^{m_{\max}-1} \end{vmatrix} \exp\left( -\frac{1}{4 l_B^2} \sum_{j=1}^{N-1} z_j z_j^* \right)$$

$$= \left( \prod_{k=1}^{N-1} z_k \right) \Psi(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_{N-1})$$

In contrast, an additional fermion to a filled Landau level must occupy a state at a higher Landau level.

The chemical potential $\mu \equiv \dfrac{\partial U}{\partial N}$ can be interpreted as the change of energy $\Delta U$ due to the addition of one particle ($\Delta N = 1$).

Thus, if we neglect all particle-particle interactions, then for each Landau level

$$\mu(N) = \begin{cases} \varepsilon_n & \text{for} \quad 0 \le N \le m_{\max} \quad \text{or} \quad 0 \le \nu < 1 \\ \varepsilon_{n+1} & \text{for} \quad N = m_{\max} + 1 \quad \text{or} \quad \nu = 1 \end{cases}$$

Taking all levels into account, $\mu$ is discontinuous at $\nu = \text{integer}$.

At which point, the system becomes incompressible with $\kappa_N = 0$.