11.3. Fractional Quantum Hall Effects

IQHE is due to non-interacting particles becoming incompressible ($\kappa_N = 0$) at filling factor $\nu = n$, where $n = 1, 2, 3, \ldots$.

FQHE is due to interacting particles becoming incompressible at some preferred filling factors $\nu = \frac{p}{q}$, where q is a odd integer.

The Laughlin Wavefunctions

Ref: R.B.Laughlin, Phys.Rev.Lett. 50,1395 (1983)

The Laughlin wavefunction for N interacting Fermions in the lowest Landau level is given by

$$\psi_L = \langle z_1, \ldots, z_N \mid 0, L \rangle = \mathcal{N} \prod_{i<j} (\tilde{z}_i - \tilde{z}_j)^m \exp\left(-\frac{1}{4} \sum_{k=1}^{N} |\tilde{z}_k|^2 \right)$$

where L stands for Laughlin,

$$\tilde{z} = \tilde{r}\, e^{i\theta} = \frac{r}{l_B} e^{i\theta} \qquad\qquad l_B^2 = \frac{\hbar c}{|e|\, B}$$

$\mathcal{N}$ is a normalization constant & m is a non-negative odd integer so that ψ_L is antisymmetric under particle exchange.

$$\psi_L = 0 \qquad \text{whenever} \qquad \tilde{z}_i = \tilde{z}_j$$

→ Particles are kept apart.

In the lowest Landau level, the 1-particle state (see 10.3._SymmetricGauge.pdf)

$$\Psi_{0\,m}(\boldsymbol{r}) = \frac{1}{\sqrt{m!\, 2\pi\, l_B^2}} \left(\frac{\tilde{z}}{\sqrt{2}} \right)^m \exp\left(-\frac{1}{4} \tilde{z}\, \tilde{z}^* \right)$$

has an angular momentum of $m\hbar$. The factor $(\tilde{z}_i - \tilde{z}_j)^m$ thus indicates a relative angular momentum of $m\hbar$ between each pair of particles. On the other hand, expanding the product $\prod_{i<j} (\tilde{z}_i - \tilde{z}_j)^m$ indicates that the 1-particle angular momentum ranges from 0 to $(N-1)\, m\hbar$.

$$|\psi_L|^2 = |\mathcal{N}|^2 \prod_{i<j} |\tilde{z}_i - \tilde{z}_j|^{2m} \exp\left[-\frac{1}{2} \sum_k |\tilde{z}_k|^2 \right]$$

$$= |\mathcal{N}|^2 \exp\left[2m \sum_{i<j} \ln |\tilde{z}_i - \tilde{z}_j| - \frac{1}{2} \sum_k |\tilde{z}_k|^2 \right]$$

$$= |\mathcal{N}|^2 \exp\left[\frac{1}{m} \left\{ 2m^2 \sum_{i<j} \ln |\tilde{z}_i - \tilde{z}_j| - \frac{1}{2} m \sum_i |\tilde{z}_i|^2 \right\} \right]$$

Consider a (quasi-) 2-D one component plasma of charge e in a uniform background charge of density ρ. (Quasi-2-D means everything is independent of the position along the direction perpendicular to a plane.)

Let the plane be the xy-plane.

By definition, a "point" charge on the plane is actually a line charge parallel to the z-axis.

Using the Gauss' law, the electric field due to a "point" charge at the origin is radial & given by

$$\boldsymbol{E}_p = E_p \hat{\boldsymbol{r}} = \frac{2q}{r} \hat{\boldsymbol{r}}$$

$$\rightarrow \qquad v_p = -\int d\boldsymbol{r} \cdot \boldsymbol{E}_p = -2q \ln \frac{r}{r_0}$$

where q is the charge per unit length along z, r_0 is some reference point where $v = 0$. The total particle-particle coulomb interaction energy (per unit length along z) is therefore

$$V_{pp} = -2q^2 \sum_{i<j} \ln \frac{|\boldsymbol{r}_i - \boldsymbol{r}_j|}{r_0}$$

Similarly, the electric field due to the background charge is

$$\boldsymbol{E}_b = E_b \hat{\boldsymbol{r}} = 2\pi \rho r \hat{\boldsymbol{r}} \qquad \text{with} \qquad \rho = -\frac{N_b q}{\pi R_0^2}$$

$$\rightarrow \qquad v_b = -\pi \rho r^2 = N_b q \left(\frac{r}{R_0}\right)^2$$

where $N_b q$ is the total background charge per unit length, R_0 is the radius of the system & we've set $v_b(0) = 0$.

The total particle-background coulomb interaction energy per unit length is therefore

$$V_{pb} = N_b q^2 \sum_i \left(\frac{r_i}{R_0}\right)^2$$

The Boltzmann factor is

$$\exp\{-\beta(V_{pp} + V_{pb})\} = \exp\left\{\beta\left[2q^2 \sum_{i<j} \ln \frac{|\boldsymbol{r}_i - \boldsymbol{r}_j|}{r_0} - N_b q^2 \sum_i \left(\frac{r_i}{R_0}\right)^2\right]\right\}$$

Comparing this with

$$\exp\left[\frac{1}{m}\left\{2m^2 \sum_{i<j} \ln |\tilde{z}_i - \tilde{z}_j| - \frac{1}{2} m \sum_i |\tilde{z}_i|^2\right\}\right]$$

$$= \exp\left[\frac{1}{m e^2}\left\{2m^2 e^2 \sum_{i<j} \ln |\tilde{z}_i - \tilde{z}_j| - \frac{1}{2} m e^2 \sum_i \left(\frac{r_i}{l_B}\right)^2\right\}\right]$$

where e is some unit of charge, we have

$$\beta = \frac{1}{m e^2} \qquad q = m e$$

$$r_0 = l_B \qquad N_b = \frac{m e^2 R_0^2}{2 q^2 l_B^2} = \frac{R_0^2}{2 m l_B^2}$$

The background charge density is therefore

$$\sigma = \frac{N_b q}{\pi R_0^2} = \frac{q}{m(2\pi l_B^2)} = \frac{e}{2\pi l_B^2}$$

Since each Landau state occupies an area of $2\pi l_B^2$, the number of states in each Landau level is

$$N_L = \frac{\pi R_0^2}{2\pi l_B^2} = \frac{R_0^2}{2 l_B^2}$$

Assuming only the lowest level is occupied ($N \le N_L$), the filling fraction for ψ_L is therefore

$$\nu = \frac{N}{N_L} = \frac{N_b}{N_L} = \frac{1}{m} \qquad (m = \text{odd})$$

where we've used the fact that, for a neutral system, $N = N_b$.

Laughlin then calculated numerically the exact solutions for a 3-particle cluster & found that the agreement with ψ_L was over 99.9 % for three types of inter-particle potentials $\frac{1}{r}$, $\ln r$ & $e^{-r^2/2}$. Assuming ψ_L is a reasonable approximation even for large N, the series $\nu = \frac{1}{m}$ of FQHE is thus explained. The same mechanism should also applies to series of the form $\nu = n + \frac{1}{m}$ by invoking additional filling of higher Landau levels.

Alternative Derivation

In the symmetric gauge, the angular momentum is given by (see 10.3._SymmetricGauge.pdf)

$$L_3 \mid n, m\rangle = e\,\hbar\,(m-n) \mid n, m\rangle$$

If we restrict ourselves to the lowest ($n = 0$) Landau level,

$$L_3 \mid 0, m\rangle = e\,\hbar\,m \mid 0, m\rangle$$

then m denotes the angular momentum.

As mentioned afore, expanding the product $\prod_{i<j} (\tilde{z}_i - \tilde{z}_j)^m$ in the Laughlin wave function indicates that the 1-particle angular momentum ranges from 0 to $(N-1)\,m\,\hbar$. The total angular momentum is therefore

$$L_{\text{total}} = m\,\hbar \sum_{n=0}^{N-1} n = \frac{1}{2} N(N-1)\,m\,\hbar$$

On the other hand, from 10.3._SymmetricGauge.pdf, radius of the orbit with angular momentum $L_3 = m\,\hbar$ is $r = \sqrt{2m}\ l_B$. Since

$$L_{\max} = (N-1)\,m\,\hbar$$

radius of the largest orbit is

$$r_{\max}^2 = 2\,(N-1)\,m\,l_B^2$$

Area occupied by ψ_L is therefore

$$S = \pi\,r_{\max}^2 = 2\,\pi\,(N-1)\,m\,l_B^2$$

Each Landau state occupies an area $\Delta S = 2\,\pi\,l_B^2$. The filling fraction for ψ_L is

$$\nu = \frac{N\,\Delta S}{S} = \frac{N}{m\,(N-1)} \simeq \frac{1}{m} \qquad \text{for } N >> 1$$