D. Groups SU (N) & SU (2 N)

Refs: Z.Q.Ma, X.Y.Gu, “Problems & Solutions in Group Theory for Physics”, 2004, §8.1.

SU(N) has N^2-1 generators that, in its self-representation, are $N\times N$ traceless hermitian matrices satisfying

$$[\lambda_A, \lambda_B] = 2i f_{AB}^C \lambda_C$$

$$\{\lambda_A, \lambda_B\} = \frac{4}{N}\delta_{AB} + 2 d_{ABC}\lambda_C$$

& $$\mathrm{Tr}(\lambda_A \lambda_B) = 2\delta_{AB}$$

where f_{AB}^C is the structural constant. f_{AB}^C & d_{ABC} are, respectively, anti-symmetric & symmetric w.r.t. interchanges in their indices.

Note: In Ma’s notation, $\lambda_A = 2T_A$.

The generators λ_A can be calculated using eq. 8.1 in Ma.

The resultant f_{AB}^C & d_{ABC} are then obtained by calculating the (anti-) commutaors.

For example, the result for SU(3) is listed in Table 8.1 of Ma.

Alternatively, one can use following trick.

Let λ_A^N be the $N\times N$ generators for SU(N). Then

$$\lambda_A^N = \begin{pmatrix} \lambda_A^{N-1} & 0 \\ 0 & 0 \end{pmatrix} \text{for } A = 1, \ \ldots, \ (N-1)^2 - 1$$

For the $2(N-1)$ λ_A^N’s labeled by $A = (N-1)^2, \ \ldots, \ N^2-2$, the only non-zero elements are given by “grafting” τ_1 & τ_2 onto the N^{th} row & column.

To be more specific, for $A = (N-1)^2, \ (N-1)^2+2, \ \ldots$ & $\quad i = \dfrac{A-(N-1)^2}{2} + 1$

$$(\lambda_A^N)_{N,i} = (\tau_1)_{21} \qquad (\lambda_A^N)_{i,N} = (\tau_1)_{12}$$

For $A = (N-1)^2+1, \ (N-1)^2+3, \ \ldots$ & $\quad i = \dfrac{A-(N-1)^2-1}{2} + 1$

$$(\lambda_A^N)_{N,i} = (\tau_2)_{21} \qquad (\lambda_A^N)_{i,N} = (\tau_2)_{12}$$

Finally, for $A = N^2-1$,

$$\lambda_A^N = \alpha\,\mathrm{diag}(1, \ 1, \ \ldots, \ 1, \ -(N-1)),$$

where α is obtained from

$$\mathrm{Tr}(\lambda_A^N \lambda_A^N) = 2$$

→ $$\alpha^2\left(N-1+(N-1)^2\right) = 2$$

i.e., $$\alpha = \sqrt{\frac{2}{N(N-1)}}$$

SU (2)

$$f_{AB}^C = \varepsilon_{ABC} \qquad d_{ABC} = 0 \qquad \lambda_A = \tau_A$$

→ $$[\lambda_A, \lambda_B] = 2i\varepsilon_{ABC}\lambda_C$$

$$\{\lambda_A, \lambda_B\} = 2\delta_{AB}$$

& $$\mathrm{Tr}(\lambda_A \lambda_B) = 2\delta_{AB}$$

$$\lambda_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

SU(3)

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

SU(4)

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_9 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad \lambda_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \lambda_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_{10} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \qquad \lambda_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \qquad \lambda_{14} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

$$\lambda_{15} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$