L. 1-D Soliton Solutions

Consider a 1-D soliton described by

$$\mathcal{H} = \frac{1}{2} f \left[\left(\frac{d\phi}{dx} \right)^2 + 2U(\phi) \right] \qquad \left(\text{Ezawa: } f = \hbar c \,,\, U = \frac{1}{\hbar c} U_{\text{Ezawa}} \right)$$

with E-L eq.

$$\frac{d^2\phi}{dx^2} - \frac{dU}{d\phi} = 0$$

$$\frac{d\phi}{dx}\frac{d^2\phi}{dx^2} = \frac{1}{2}\frac{d}{dx}\left(\frac{d\phi}{dx}\right)^2 \qquad \frac{d\phi}{dx}\frac{dU}{d\phi} = \frac{dU}{dx}$$

$$\rightarrow \quad \frac{1}{2}\frac{d}{dx}\left(\frac{d\phi}{dx}\right)^2 - \frac{dU}{dx} = 0$$

$$\therefore \quad \frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 - U + u_0 = 0 \qquad (u_0 = \text{const})$$

$$\frac{d\phi}{dx} = \pm\sqrt{2(U-u_0)}$$

$$\rightarrow \quad x = \pm \int \frac{d\phi}{\sqrt{2(U-u_0)}} + c_\pm \qquad (c_\pm = \text{constant})$$

or

$$x - x_0 = \pm \int_{\phi(x_0)}^{\phi(x)} \frac{d\phi}{\sqrt{2(U-u_0)}}$$

&

$$\mathcal{H} = f\left[2U(\phi) - u_0\right]$$

Kinks for Real K-G ϕ^4 Field

For the real K-G ϕ^4 field, (see 7.3._SolitaryWavesKinksAndSolitons.pdf)

$$\mathcal{H} = \frac{1}{2} f \left[(\partial_x \phi)^2 + \frac{g}{2}(\phi^2 - v^2)^2 \right]$$

$$\rightarrow \quad U(\phi) = \frac{g}{4}(\phi^2 - v^2)^2$$

For kinks, boundary conditions are $\phi(\pm\infty) = \pm v$.

$$\therefore \quad U(\pm\infty) = 0$$

$$\left.\frac{d\phi}{dx}\right|_{x\to\pm\infty} = \pm\sqrt{2(-u_0)} = 0 \qquad \rightarrow \qquad u_0 = 0$$

Let

$$\eta = \phi^2 - v^2$$

$$\rightarrow \quad \phi = \pm\sqrt{\eta + v^2} \qquad d\phi = \pm \frac{d\eta}{2\sqrt{\eta + v^2}}$$

$$U(\eta) = \frac{g}{4}\eta^2$$

To avoid the proliferation of ± signs, we'll work only with the + sign & consider the – case only at the end.

$$\rightarrow \qquad x = \int \frac{d\phi}{\sqrt{2U}} + c = \sqrt{\frac{1}{2g}} \int \frac{d\eta}{\eta\sqrt{\eta + v^2}} + c$$

$$\int \frac{dx}{x\sqrt{x+a}} = -\frac{2}{\sqrt{a}} \tanh^{-1} \frac{\sqrt{x+a}}{\sqrt{a}}$$

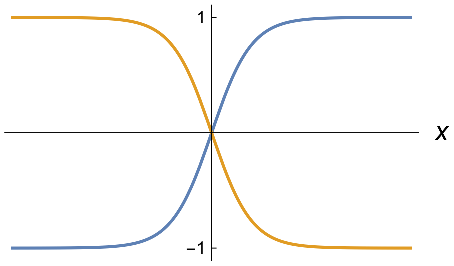
$$\rightarrow \qquad x = -\sqrt{\frac{2}{g}} \frac{1}{v} \tanh^{-1}\left(\frac{\sqrt{\eta + v^2}}{v}\right) + c$$

$$\sqrt{\eta + v^2} = -v \tanh\left[v \sqrt{\frac{g}{2}} (x - c)\right]$$

$$\therefore \qquad \phi = \pm\sqrt{\eta + v^2}$$

$$\rightarrow \qquad \phi = \mp v \tanh\left[v \sqrt{\frac{g}{2}} (x - c)\right]$$

tanh(x),–tanh(x)



Since $\tanh 0 = 0$, we can write

$$\phi(x) - \phi(x_0) = \pm v \tanh\left[v \sqrt{\frac{g}{2}} (x - x_0)\right]$$

Using $\tanh(\pm\infty) \rightarrow \pm\infty$, we see that the +(–) sign denotes kink (anti-kink).

Setting $\xi = \frac{1}{v}\sqrt{\frac{1}{2g}}$

we have

$$\phi(x) - \phi(x_0) = \pm v \tanh\left(\frac{x - x_0}{2\xi}\right)$$

so that ξ is a measure of the width of the transition region where ϕ changes significantly.

Incidentally, setting $\eta^2 = \phi^2 - v^2$ leads to a solution of the form

$$\phi(x) = \pm v \coth\left(\frac{x - c}{2\xi}\right)$$

However, since $\lim_{x \to 0^\pm} \coth x = \pm\infty$, it can't be used for our purposes.

Without loss of generality, we can set $x_0 = 0$ so that

$$\phi(x) = \pm v \tanh\left(\frac{x}{2\xi}\right)$$

$$\rightarrow \qquad \phi^2 - v^2 = v^2\left[\tanh^2\left(\frac{x}{2\,\xi}\right) - 1\right] = -v^2\,\mathrm{sech}^2\left(\frac{x}{2\,\xi}\right)$$

$$\partial_x \phi = \pm\,\frac{v}{2\,\xi}\,\mathrm{sech}^2\left(\frac{x}{2\,\xi}\right)$$

$$\therefore \qquad \mathcal{H} = \frac{1}{2} f\left[(\partial_x \phi)^2 + \frac{g}{2}(\phi^2 - v^2)^2\right]$$

$$= \frac{1}{2} f\left[\left(\frac{v}{2\,\xi}\right)^2 + \frac{g}{2} v^4\right]\mathrm{sech}^4\left(\frac{x}{2\,\xi}\right)$$

$$= \frac{1}{2} f\, g\, v^4\,\mathrm{sech}^4\left(\frac{x}{2\,\xi}\right)$$

Kinks for Sine-Gordon Field

See 7.4._Sine-GordonSolitons.pdf .

$$\mathcal{H} = \frac{1}{2} f\left[(\partial_x \phi)^2 + \frac{2}{\lambda^2}(1 - \cos\phi)\right] \qquad \left(\text{Ezawa: } f = \hbar c\,,\ U = \frac{1}{\hbar c} U_{\text{Ezawa}}\right)$$

$$\rightarrow \qquad U(\phi) = \frac{1}{\lambda^2}(1 - \cos\phi) = \frac{2}{\lambda^2}\sin^2\frac{\phi}{2}$$

$$x = \pm\int\frac{d\,\phi}{\sqrt{2\,(U - u_0)}} + c_\pm$$

$$= \pm\,\frac{\lambda}{2}\int\frac{d\,\phi}{\sqrt{\sin^2\frac{\phi}{2} - \frac{1}{2}\lambda^2 u_0}} + c_\pm$$

$$= \pm\,\frac{\lambda}{2}\int\frac{d\,\phi}{\sqrt{1 - \frac{1}{2}\lambda^2 u_0 - \cos^2\frac{\phi}{2}}} + c_\pm$$

Let $\kappa^2 = \dfrac{1}{1 - \frac{1}{2}\lambda^2 u_0}$

$$\rightarrow \qquad x = \pm\,\frac{\lambda}{2}\,\kappa\int\frac{d\,\phi}{\sqrt{1 - \kappa^2\cos^2\frac{\phi}{2}}} + c_\pm$$

Note that the integral is periodic with period $2\,\pi$, which means that if $\phi(x)$ is a solution, so are

$\phi(x) + 2\,\pi\,n$ for $n = 0, \pm 1, \pm 2, \ldots$

Since x is real, we must have $\kappa^2 \leq 1$ if ϕ is unrestricted.

Another way to write the solution is

$$x - x_0 = \pm\,\frac{\lambda}{2}\,\kappa\int_{\phi(x_0)}^{\phi(x)}\frac{d\,\phi}{\sqrt{1 - \kappa^2\cos^2\frac{\phi}{2}}}$$

S-G Kinks

$$\lim_{x\to\infty} \phi(x) = 0$$

$$\to \qquad \infty = \frac{\lambda}{2}\kappa \int_{\phi(x_0)}^{0} \frac{d\phi}{\sqrt{1-\kappa^2\cos^2\frac{\phi}{2}}}$$

which is possible only if the integrand has a pole at $\phi = 0$, i.e.,

$$\kappa^2 = 1 \qquad \leftrightarrow \qquad u_0 = 0$$

so that

$$x = \pm\frac{\lambda}{2}\int \frac{d\phi}{\sin\frac{\phi}{2}} + c_\pm \quad = \pm\lambda \int \frac{d\phi}{\sin\phi} + c_\pm$$

where we've set $\kappa = 1$ without lost of generality.

Note that the integrand now has poles at $\phi = n\pi$, for $n = 0, \pm 1, \ldots$.

Using

$$\int \frac{d\phi}{\sin\phi} = \ln\left|\tan\frac{\phi}{2}\right|$$

we have

$$x = \pm\lambda \ln\left|\tan\frac{\phi}{2}\right| + c_\pm$$

$$\to \qquad \phi_\pm(x) = 2\tan^{-1}\left[\exp\left(\pm\frac{x-c_\pm}{\lambda}\right)\right]$$

Using

$$\tan^{-1} 1 = \frac{\pi}{4} \quad , \qquad \tan^{-1} 0 = 0 \qquad \& \qquad \tan^{-1}(\infty) = \frac{\pi}{2}$$

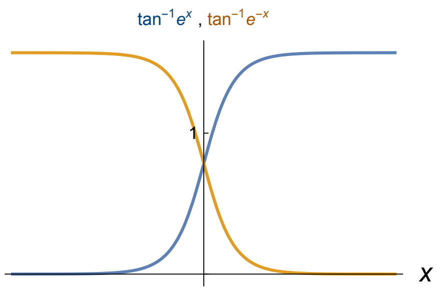
we can write

$$\phi_\pm(x) = \phi(x_0)\,\frac{4}{\pi}\tan^{-1}\left[\exp\left(\pm\frac{x-x_0}{\lambda}\right)\right]$$

where

$$\phi_+(\pm\infty) = \begin{cases} 2\,\phi(x_0) \\ 0 \end{cases} \qquad\qquad \phi_-(\pm\infty) = \begin{cases} 0 \\ 2\,\phi(x_0) \end{cases}$$

Thus, $\phi_{+/-}$ denotes a kink / anti-kink.



Kink & anti-kink

As mentioned before,

$$\phi_{+/-} + 2\pi n \qquad \text{with} \quad n = 0, \pm 1, \pm 2 \ldots.$$

is also a legitimate kink / anti-kink. On the other hand, the height of a kink / anti-kink is $2\,\phi(x_0)$.
Thus, in order to have multiple kinks / anti-kinks co-existing in the system, the height must equal to

the period, i.e.,

$$\phi(x_0) = \pi$$

By definition, only such kinks / anti-kinks can be called solitons.

A wave that vanishes at both $x \to \pm\infty$ would be

$$\phi(x) = \phi(x_0)\, \frac{4}{\pi} \tan^{-1}\left[\exp\left(-\frac{|\, x - x_0 \,|}{\lambda} \right) \right]$$

Unfortunately, since $\partial_x \phi$ is not defined at $x = x_0$, it is not admissible as a solution.

$\kappa \to 0$

For $\kappa \to 0$,

$$x = \pm\left[\frac{\lambda}{2} \kappa \int \frac{d\phi}{\sqrt{1 - \kappa^2 \cos^2 \frac{\phi}{2}}} + c_\pm \right]$$

$$\to \pm\left[\frac{\lambda}{2} \kappa\, \phi(x) + c_\pm \right]$$

$\therefore \qquad \phi(x) = \pm \dfrac{2}{\kappa \lambda} (x \mp c_\pm)$

or $\qquad \phi(x) - \phi(x_0) = \pm \dfrac{2}{\kappa \lambda} (x - x_0)$

$\mathcal{H}$

For a kink centered at the origin with $\phi(0) = (2n+1)\pi$ & rises from $2n\pi$ to $2(n+1)\pi$:

$$\phi(x) = 4 \tan^{-1}\left[\exp\left(\frac{x}{\lambda} \right) \right] + 2n\pi$$

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$$

$\to \qquad \partial_x \phi = \dfrac{4}{\lambda} \dfrac{e^{x/\lambda}}{1 + e^{2x/\lambda}} = \dfrac{2}{\lambda} \operatorname{sech} \dfrac{x}{\lambda} \qquad (\partial_x \phi)^2 = \dfrac{16}{\lambda^2} \dfrac{e^{2x/\lambda}}{(1 + e^{2x/\lambda})^2} = \dfrac{4}{\lambda^2} \operatorname{sech}^2 \dfrac{x}{\lambda}$

$$\tan \frac{\phi}{4} = \frac{e^{x/\lambda} + \tan(n \frac{\pi}{2})}{1 - e^{x/\lambda} \tan(n \frac{\pi}{2})} = \begin{cases} e^{x/\lambda} & \text{for } n = \text{even} \\ -e^{-x/\lambda} & \text{for } n = \text{odd} \end{cases}$$

$$1 - \cos\phi = 2 \sin^2 \frac{\phi}{2} = \frac{2 \tan^2 \frac{\phi}{2}}{1 + \tan^2 \frac{\phi}{2}}$$

Using

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

we see that

$$\tan \frac{\phi}{2} = \begin{cases} \dfrac{2 e^{x/\lambda}}{1 - e^{2x/\lambda}} & \text{for } n = \text{even} \\ -\dfrac{2 e^{-x/\lambda}}{1 - e^{-2x/\lambda}} & \text{for } n = \text{odd} \end{cases} = \frac{2 e^{x/\lambda}}{1 - e^{2x/\lambda}} = -\operatorname{csch} \frac{x}{\lambda}$$

$$\rightarrow \qquad 1+\tan^2\frac{\phi}{2}=\frac{\left(1+e^{2x/\lambda}\right)^2}{\left(1-e^{2x/\lambda}\right)^2}=\coth^2\frac{x}{\lambda}$$

$$\therefore \qquad 1-\cos\phi=\frac{8\,e^{2x/\lambda}}{\left(1+e^{2x/\lambda}\right)^2}=2\,\operatorname{sech}^2\frac{x}{\lambda}$$

Hence

$$\mathcal{H}=\frac{1}{2}f\left[(\partial_x\phi)^2+\frac{2}{\lambda^2}(1-\cos\phi)\right]$$

$$=\frac{16}{\lambda^2}f\,\frac{e^{2x/\lambda}}{\left(1+e^{2x/\lambda}\right)^2}=\frac{4}{\lambda^2}f\,\operatorname{sech}^2\frac{x}{\lambda}$$

$$E=\frac{16}{\lambda^2}f\int_{-\infty}^{\infty}dx\,\frac{e^{2x/\lambda}}{\left(1+e^{2x/\lambda}\right)^2}$$

$$=\frac{8}{\lambda}f\int_0^{\infty}dy\,\frac{1}{(1+y)^2}$$

$$=\frac{8}{\lambda}f\left(-\frac{1}{1+y}\right)_0^{\infty}$$

$$=\frac{8}{\lambda}f$$

E-L eq., Static Case

E-L eq. for the S-G field is

$$\frac{1}{c^2}\partial_{tt}\phi-\partial_{xx}\phi+\frac{1}{\lambda^2}\sin\phi=0$$

Consider the static kink (see section $\mathcal{H}$)

$$\phi(x)=4\tan^{-1}\left[\exp\left(\frac{x}{\lambda}\right)\right]+2\pi n$$

$$\frac{d\,\operatorname{sech}x}{dx}=-\operatorname{sech}x\tanh x$$

$$\partial_x\phi=\frac{2}{\lambda}\operatorname{sech}\frac{x}{\lambda} \qquad \rightarrow \qquad \partial_{xx}\phi=-\frac{2}{\lambda^2}\operatorname{sech}\frac{x}{\lambda}\tanh\frac{x}{\lambda}$$

$$1-\cos\phi=2\,\operatorname{sech}^2\frac{x}{\lambda}$$

$$\rightarrow \qquad \sin\phi\,\frac{d\phi}{dx}=-\frac{4}{\lambda}\operatorname{sech}^2\frac{x}{\lambda}\tanh\frac{x}{\lambda}$$

$$\therefore \qquad \sin\phi=-2\,\operatorname{sech}\frac{x}{\lambda}\tanh\frac{x}{\lambda}$$

Together with $\partial_t\phi=0$, we see that $\phi(x)$ does satisfy the E-L eq.

E-L eq.

The time-dependent solution is obtained by a Lorentz boost of the static solution in the rest frame to a frame moving with velocity u. Hence

$$\phi(t,\ x) = 4\tan^{-1}\left[\exp\left(\frac{x - u\,t}{\lambda\sqrt{1 - u^2/c^2}}\right)\right] + 2\pi n$$

With $X = \dfrac{x - u\,t}{\lambda\sqrt{1 - u^2/c^2}}$, we have

$$\phi(t,\ x) = 4\tan^{-1}\left(e^X\right) + 2\pi n$$

$$\partial_x\phi = \frac{2}{\lambda\sqrt{1 - u^2/c^2}}\ \operatorname{sech} X$$

$$\partial_{x\,x}\phi = -\frac{2}{\lambda^2\left(1 - u^2/c^2\right)}\ \operatorname{sech} X \tanh X$$

$$\sin\phi = -2\ \operatorname{sech} X \tanh X$$

& $$\partial_t\phi = -\frac{2\,u}{\lambda\sqrt{1 - u^2/c^2}}\ \operatorname{sech} X$$

$$\partial_{t\,t}\phi = -\frac{2\,u^2}{\lambda^2\left(1 - u^2/c^2\right)}\ \operatorname{sech} X \tanh X$$

Thus, $\phi(t,\ x)$ does satisfy the E-L eq.

$$\frac{1}{c^2}\,\partial_{t\,t}\phi - \partial_{x\,x}\phi + \frac{1}{\lambda^2}\sin\phi = 0$$

Similarly, with

$$1 - \cos\phi = 2\ \operatorname{sech}^2 X$$

we have

$$\mathcal{H} = \frac{1}{2} f\left[(\partial_x\phi)^2 + \frac{2}{\lambda^2}(1 - \cos\phi)\right]$$

$$= \frac{1}{2} f\left[\frac{4}{\lambda^2\left(1 - u^2/c^2\right)}\ \operatorname{sech}^2 X + \frac{4}{\lambda^2}\operatorname{sech}^2 X\right]$$

$$= 2f\,\frac{2 - u^2/c^2}{\lambda^2\left(1 - u^2/c^2\right)}\operatorname{sech}^2 X$$

$$E = 2f\,\frac{2 - u^2/c^2}{\lambda^2\left(1 - u^2/c^2\right)}\int_{-\infty}^{\infty} d\,x\,\operatorname{sech}^2 X$$

$$= 2f\,\frac{2 - u^2/c^2}{\lambda\sqrt{1 - u^2/c^2}}\int_{-\infty}^{\infty} d\,X\ \operatorname{sech}^2 X$$

$$\int d\,x\,\operatorname{sech}^2 x = \tanh x \qquad\qquad \tanh(\pm\infty) = \pm 1$$

→ $$E = 4f\,\frac{2 - u^2/c^2}{\lambda\sqrt{1 - u^2/c^2}}$$