4. Elementary Probability Theory And Limit Theorems

4.A. Introductory Remarks

To begin, we introduce an intuitive definition of **probability** that requires only the counting of ways to satisfy some given criteria. This will be justified later by the **law of large numbers**. Next, we deal with **stochastic variables** whose values can only be specified in terms of probabilities. Basic concepts and tools discussed include **probability distributions**, **moments** of stochastic variables, and **characteristic functions**. As examples, detailed discussions are given on the **binomial**, **Gaussian** and **Poisson** distributions. Finally, we introduce an elementary form of the **central limit theorem** and use it to prove the **law of large numbers**.

In the special topics sections, the **random walk** problems in 1-, 2-, and 3-D are first discussed. Next, we introduce the concept of an **infinitely divisible stochastic variable** and then derive the **central limit theorem** for uncorrelated (independent) stochastic variables of finite variance. The behavior of stochastic variables of infinite variance is discussed in terms of the **Levy distributions**. Fractal- like spatial clustering in such systems is illustrated by the **Weierstrass random walk**.

4.B. Permutations & Combinations

Addition Principle

Let 2 operations be mutually exclusive.
If the number of ways of doing them is *m* and *n*, respectively, then the number of ways of doing either of them is $m+n$.

Multiplication Principle

Let there be *n* ways to perform an operation.
After which, let there be *m* ways to perform another operation.
Then, these 2 operations can be done in $n \times m$ ways.

Permutation

A **permutation** is any arrangement of a set of distinct objects in a definite order.

The number of different permutations of *N* distinct objects is $N!$.

The number of different permutations of any *R* objects taken from a set of *N* is

$$P_R^N = N(N-1)\cdots(N-R+1) = \frac{N!}{(N-R)!}.$$

Combination

A **combination** is a selection out of a set of distinct objects with no regard to order.

The number of different combinations of any *R* objects taken from a set of *N* is

$$C_R^N = \frac{N(N-1)\cdots(N-R+1)}{R!} = \frac{N!}{(N-R)!R!}$$

Consider a set of *k* different kinds of objects.

Let n_j be the number of the *j*th kind of objects, and *N* the total number of objects.

By definition, $N = \sum_{j=1}^{k} n_j$.

The number of different permutations of such a set is

$$\frac{N!}{n_1!n_2!\cdots n_k!}$$

Exercise 4.1

1. Find the number of permutations of the letters in “ENGINEERING”.
2. In how many ways are 3 E’s together?
3. In how many ways are (only) 2 E’s together?

Answer

1. There’re 3E’s, 2G’s, 2I’s, 3N’s and 1R in this 11-letter word.

   The number of permutations is therefore $\frac{11!}{3!2!2!3!1!} = 277,200$.
2. Consider the 3E’s as a single letter ε.
   There’re 1 ε, 2G’s, 2I’s, 3N’s and 1R in the 9-letter word “εNGINRING”.

   The number of permutations is $\frac{9!}{1!2!2!3!1!} = 15,120$.
3. Consider ε as representing 2E’s.
   Given a particular permutation of “εNGINRING”, there are 8 ways to insert an E so that it is not adjacent to ε.
   The number of permutations for only 2E’s together is thus $8 \times 15,120 = 120,960$.

4.C. Definition Of Probability

A quick practical introduction is the book "Probability and Statistics" of the Schaum's Outline Series.

If out of *N* identical experiments, N_A of them result in event *A*.

The probability $P(A)$ of *A* occurring is defined as

$$P(A) = \lim_{N \to \infty} \frac{N_A}{N}.$$

Alternatively, if an experiment can result in *n* different equally probable outcomes and *m* of them corresponds to event *A*, then

$$P(A) = \frac{m}{n}$$

A **sample space** of an experiment is the set *S* that contains all possible outcomes. In other words, any outcome of the experiment must correspond to one or more elements of *S*. An **event** is a subset of *S*.

The probability of an event *A* can be found using the following procedure:

a) Set up a sample space *S*.

b) Assign probabilities to all sample points.
For the special case of *N* equally likely outcomes, each sample point has a probability $\frac{1}{N}$.

c) The probability of an event *A* is the sum of the probabilities of all the sample points included in the event.

By definition,

$$P(\varnothing) = 0, \qquad P(S) = 1$$

where $\varnothing$ is the empty set.

Consider 2 events *A* and *B* with probabilities $P(A)$ and $P(B)$, respectively.

1. They are **mutually exclusive** if $A \cap B = \varnothing$.

2. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (4.1)

3. If A, B are mutually exclusive,

$$P(A \cup B) = P(A) + P(B) \tag{4.2}$$

4. If events $\{A_j\}$, with $j = 1, \cdots, m$, are mutually exclusive and **exhaustive**, then

$$\bigcup_{j=1}^{m} A_j = S$$

and $\{A_j\}$ forms a **partition** of S into m subsets, whereupon,

$$\sum_{j=1}^{m} P(A_j) = 1 \tag{4.3}$$

5. Events A and B are **independent** if and only if

$$P(A \cap B) = P(A)P(B) \tag{4.4}$$

6. For **mutually exclusive** events,

$$P(A \cap B) = P(\varnothing) = 0$$

7. The **conditional probability** $P(B \mid A)$ is the probability of A given B. Hence

$$P(A \cap B) = P(B)P(B \mid A) \tag{4.5}$$

Alternatively, $P(B \mid A)$ is the probability of A using B as the sample space.

8. Since

$$P(A \cap B) = P(B \cap A)$$

we have

$$P(B)P(B \mid A) = P(A)P(A \mid B) \tag{4.6}$$

9. If A, B are independent,

$$P(B \mid A) = \frac{P(A \cap B)}{P(B)} = P(A) \tag{4.7}$$

Note

1. Independent events are not mutually exclusive.

Exercise 4.2

Let $P(A) = \frac{3}{5}$, $P(B) = \frac{2}{3}$, and $P(A \cup B) = 1$.

Compute $P(A \cap B)$, $P(B \mid A)$, and $P(B \mid A)$.

Are *A* and *B* independent?

Answer

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

$$= \frac{3}{5} + \frac{2}{3} - 1 = \frac{9 + 10 - 15}{15} = \frac{4}{15}$$

$$P(A)P(B) = \frac{3}{5} \cdot \frac{2}{3} = \frac{6}{15} \neq P(A \cap B) \quad \Rightarrow \quad A \text{ and } B \text{ not independent.}$$

$$P(A \mid B) = \frac{P(A \cap B)}{P(A)} = \frac{4}{15} \cdot \frac{5}{3} = \frac{4}{9}$$

$$P(B \mid A) = \frac{P(B \cap A)}{P(B)} = \frac{4}{15} \cdot \frac{3}{2} = \frac{2}{5}$$

4.D. Stochastic Variables And Probabilities

A **stochastic**, or **random**, **variable** is a quantity whose value is determined by the outcome of an experiment.

A stochastic variable X on a sample space S is a 1-1 onto function

$$X : S \to R$$

where R is a subset of the real numbers.
Thus, X assigns a real number to every sample point.

The possible values of X are denoted by $\{x_i\}$.

4.D.1. Distribution Functions

4.D.1.a. Discrete Stochastic Variables

Let the realizations of the stochastic variable X on S be a countable set $\{x_i\}$.

S becomes a probability space if a probability p_i is assigned to each x_i.

The set $\{p_i\}$ is called a **probability distribution** on S.

By definition, it must be

1. Positive-definite: $p_i \ge 0$
2. Normalized: $\sum_i p_i = 1$

In order to make use of the powerful tools of mathematical analysis, we need to convert all variables and functions into their continuous analogues.

Now, a **probability density function** $P_X(x)$ is defined by

$$P_X(x)\,dx = \text{ probability of finding } X \in \left(x - \frac{1}{2}dx, x + \frac{1}{2}dx \right)$$

so that for any finite interval $[a,b]$,

$$\text{Prob}(a \le x \le b) = \int_a^b dx P_X(x) \qquad (4.10)$$

Note that x can always be taken to range over the entire real axis.

Thus, given $\{p_i\}$, we can define

$$P_X(x) = \sum_{i=1}^{n} p_i \delta(x - x_i) \qquad (4.8)$$

so that for Δx smaller than the minimum spacing of x_i,

$$\int_{x_j - \Delta x}^{x_j + \Delta x} dx P_X(x) = \sum_{i=1}^{n} p_i \int_{x_j - \Delta x}^{x_j + \Delta x} dx \delta(x - x_i) = \sum_{i=1}^{n} p_i \delta_{ij} = p_j$$

Furthermore,

$$\int_{-\infty}^{\infty} dx P_X(x) = \sum_{i=1}^{n} p_i = 1$$

Another useful quantity is the accumulate distribution function $F_X(x)$ defined as

$$F_X(x) = \int_{-\infty}^{x} dy P_X(y)$$

$$= \sum_{i=1}^{n} p_i \int_{-\infty}^{x} dy \delta(y - x_i)$$

$$= \sum_{i=1}^{n} p_i \Theta(x - x_i) \qquad (4.9)$$

where

$$\Theta(x - x_i) = \int_{-\infty}^{x} dy \delta(y - x_i) = \begin{cases} 0 & \text{for} \quad x < x_i \\ 1 & \quad\quad\; x > x_i \end{cases}$$

is a Heaviside step function.

Thus, $F_X(x)$ is the probability that $X \in (-\infty, x]$. By definition, $F_X(-\infty) = 0$.

Note that

$$\frac{dF_X(x)}{dx} = \sum_{i=1}^{n} p_i \frac{d}{dx} \Theta(x - x_i)$$

$$= \sum_{i=1}^{n} p_i \delta(x - x_i)$$

$$= P_X(x)$$

Now, the criterion $p_i \geq 0$ implies $P_X(x) \geq 0$ so that $F_X(x)$ must be monotonically increasing. The normalization of $\sum_{i=1}^{n} p_i = 1$ means that

$$F_X(\infty) = \int_{-\infty}^{\infty} dx P_X(x)$$

$$= \sum_{i=1}^{n} p_i \int_{-\infty}^{\infty} dx \delta(x - x_i) = \sum_{i=1}^{n} p_i = 1$$

Exercise 4.3

Consider a loaded die with

p_1	p_2	p_3	p_4	p_5	p_6
$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{3}{12}$

Plot $P_X(x)$ and $F_X(x)$.

4.D.1.b. Continuous Stochastic Variables

Let $X(S)$ be a continuous set, eg., an interval on the real axis.

X is called a **continuous stochastic variable**.
By definition, any interval, say, $a \le X \le b$, corresponds to an event in *S*.

Let there be a piecewise continuous function $P_X(x)$ such that the probability of *X* having a value in the interval $[a,b]$ is given by

$$\text{Prob}(a \le x \le b) = \int_a^b dx P_X(x) \qquad (4.10)$$

$P_X(x)$ is called the **probability density** of *X*. It must be

1. Positive-definite: $P_X(x) \ge 0$

2. Normalized: $\int P_X(x)\, dx = 1$

where the range of the integral can always be taken as the whole real axis.

The accumulate distribution function $F_X(x)$ defined as

$$F_X(x) = \int_{-\infty}^{x} dy P_X(y)$$

gives the probability that $X \in (-\infty, x]$. By definition, $F_X(-\infty) = 0$. Since $P_X(x) \ge 0$, $F_X(x)$ must be monotonically increasing. Normalization of $P_X(x)$ means that

$$F_X(\infty) = \int_{-\infty}^{\infty} dx P_X(x) = 1$$

Consider another stochastic variable

$$Y = H(X)$$

The probability density $P_Y(y)$ of Y is related to $P_X(x)$ by

$$P_Y(y) = \int_{-\infty}^{\infty} dx\delta\left[y - H(x)\right] P_X(x) \qquad (4.12)$$

$$= \sum_i \int_{-\infty}^{\infty} dx \frac{1}{\left|H'(x_i)\right|} \delta(x - x_i) P_X(x)$$

$$= \sum_i \frac{1}{\left|H'(x_i)\right|} P_X(x_i)$$

where x_i are roots of $y - H(x) = 0$ and $H' = \frac{dH}{dx}$.

Exercise 4.4

Plot the Gaussian $P_X(x) = \frac{1}{\sqrt{8\pi}} \exp\left(-\frac{x^2}{8}\right)$ and its accumulative distribution $F_X(x)$.

4.D.2. Moments

Note that $P_X(x)$ contains all possible information on *X*.

However, full knowledge of $P_X(x)$ is often not available in practice.

What one usually have is the values of the moments of *X*,

$$\langle x^m \rangle = \sum_i x_i^m p_i$$

$$= \int_{-\infty}^{\infty} dx x^m P_X(x) \qquad (4.13)$$

Some well-known nomenclature:

Mean: $\langle x \rangle$

Variance: $\langle x^2 \rangle - \langle x \rangle^2$

Standard Deviation: $\sigma_X = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \qquad (4.14)$

1st Moment

The 1st moment or mean $\langle x \rangle$ gives the "center of mass" of $P_X(x)$.

The most probable value x_p denotes the location of the maximum of $P_X(x)$, i.e.,

$$P_X(x_p) = \max P_X(x)$$

The median x_m denotes the value of *x* which divides the area under $P_X(x)$ into equal parts, i.e.,

$$F_X(x_m) = \frac{1}{2}$$

Exercise 4.5

Locate $\langle x \rangle$, x_p and x_m for $P_X(x)$ and $F_X(x)$ shown below.

Answer

$$P_X(x) = \begin{cases} \frac{4}{3}x + \frac{7}{3} & x \in \left[-\frac{7}{4}, -1\right] \\ -\frac{4}{3}x - \frac{1}{3} & \text{for} \quad x \in \left[-1, -\frac{1}{4}\right] \\ \frac{1}{2} & x \in \left[\frac{1}{2}, 1\right] \\ 0 & \textit{otherwise} \end{cases}$$

$$\langle x \rangle = \frac{1}{3}\int_{-7/4}^{-1} dx(4x+7)\,x - \frac{1}{3}\int_{-1}^{-1/4} dx(4x+1)\,x + \frac{1}{2}\int_{1/2}^{1} dxx$$

$$= -\frac{9}{16} = -0.5625 \qquad \text{[evaluated using Mathematica]}$$

$$x_p = -1 \quad \text{with} \qquad P_X(-1) = 1$$

$$F_X(x) = \begin{cases} 0 & x \le -\frac{7}{4} \\ \frac{1}{3}\int_{-7/4}^{x} dx(4x+7) & x \in \left[-\frac{7}{4}, -1\right] \\ \frac{3}{8} - \frac{1}{3}\int_{-1}^{x} dx(4x+1) & \text{for} \quad x \in \left[-1, -\frac{1}{4}\right] \\ \frac{3}{4} + \frac{1}{2}\int_{1/2}^{x} dx & x \in \left[\frac{1}{2}, 1\right] \\ 1 & x \ge 1 \end{cases}$$

Hence, $x_m \in \left[-1, -\frac{1}{4}\right]$ with

$$\frac{1}{2} = \frac{3}{8} - \frac{1}{3}\int_{-1}^{x_m} dx(4x+1)$$

$$= \frac{3}{8} - \frac{1}{3}\left(2x_m^2 - 2 + x_m + 1\right)$$

or

$$2x_m^2 + x_m - \frac{5}{8} = 0$$

$$x_m = \frac{1}{4}\left(-1 - \sqrt{6}\right) = -0.862372$$

2^{nd} Moment

The 2^{nd} moment $\langle x^2 \rangle$ gives the "moment of inertia" of $P_X(x)$ about $x = 0$.

The standard deviation σ_X measures the "spread" of $P_X(x)$ about $\langle x \rangle$.

For Exercise 4.5,

$$\langle x^2 \rangle = \frac{1}{3}\int_{-7/4}^{-1} dx(4x+7)x^2 - \frac{1}{3}\int_{-1}^{-1/4} dx(4x+1)x^2 + \frac{1}{2}\int_{1/2}^{1} dxx^2$$

$$= \frac{371}{384} = 0.966146 \qquad \text{[evaluated using Mathematica]}$$

$$\sigma_X = \sqrt{\frac{371}{384} - \left(\frac{9}{16}\right)^2} = \sqrt{\frac{499}{768}} = 0.806064$$

3^{rd} Moment

The 3^{rd} moment $\langle x^3 \rangle$ measures the "skewness" of $P_X(x)$ about $x = 0$.

For Exercise 4.5,

$$\langle x^3 \rangle = \frac{1}{3}\int_{-7/4}^{-1} dx(4x+7)x^3 - \frac{1}{3}\int_{-1}^{-1/4} dx(4x+1)x^3 + \frac{1}{2}\int_{1/2}^{1} dxx^3$$

$$= -\frac{27}{32} = -0.84375$$

4.D.3. Characteristic Functions

The **characteristic function** $f_X(k)$ of a stochastic variable *X* is defined by

$$f_X(k) = \langle e^{ikx} \rangle = \int_{-\infty}^{\infty} dx e^{ikx} P_X(x) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \langle x^n \rangle \qquad (4.15)$$

where the series expansion is applicable only if it converges.

Some useful properties are,

$$f_X(0) = \int dx P_X(x) = 1$$

$$|f_X(k)| = \left| \int dx e^{ikx} P_X(x) \right| \le \int dx \left| e^{ikx} P_X(x) \right| = \int dx P_X(x) = 1$$

$$f_X(-k) = \int dx e^{-ikx} P_X(x) = \left[\int dx e^{ikx} P_X(x) \right]^* = f_X(k)^*$$

Taking the inverse transform of (4.15), we have

$$P_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{-ikx} f_X(k) \qquad (4.16)$$

Thus, if all moments are known, $f_X(k)$, and hence, $P_X(x)$ are known using (4.15-6). Conversely, given $f_X(k)$, all the moments are known using

$$\langle x^m \rangle = \frac{1}{i^m} \frac{d^m f_X(k)}{dk^m} \bigg|_{k=0} \qquad (4.17)$$

The **cumulant expansion** of $f_X(k)$ is defined as

$$f_X(k) = \exp\left[\sum_{n=1}^{\infty} \frac{(ik)^n}{n!} C_n(X) \right] \qquad (4.18)$$

where $C_n(X)$ is called the *n*th order **cumulant**.

Using

$$\frac{d}{dk} f_X = f_X' = \chi f_X$$

where $\chi = \sum_{n=1}^{\infty} \frac{i^n k^{n-1}}{(n-1)!} C_n(X) = \sum_{n=0}^{\infty} \frac{i^{n+1} k^n}{n!} C_{n+1}(X)$

we have

$$\frac{d^2}{dk^2} f_X = \chi' f_X + \chi f_X' = \left(\chi' + \chi^2\right) f_X$$

$$\frac{d^3}{dk^3} f_X = \left[\chi'' + 2\chi\chi' + \left(\chi' + \chi^2\right)\chi\right] f_X = \left(\chi'' + 3\chi\chi' + \chi^3\right) f_X$$

$$\frac{d^4}{dk^4} f_X = \left[\chi''' + 3\chi\chi'' + 3\chi'^2 + 3\chi^2\chi' + \left(\chi'' + 3\chi\chi' + \chi^3\right)\chi\right] f_X$$

$$= \left(\chi''' + 4\chi\chi'' + 3\chi'^2 + 6\chi^2\chi' + \chi^4\right) f_X$$

and so on. Thus, using

$$f_X(0) = 1$$

$$\frac{d^m \chi}{dk^m} = \chi^{(m)} = \sum_{n=0}^{\infty} \frac{i^{n+m+1} k^n}{n!} C_{n+m+1}(X)$$

$$\frac{1}{i^{m+1}} \chi^{(m)}\Big|_{k=0} = C_{m+1}(X) \qquad \text{with} \qquad \frac{1}{i}\chi = \frac{1}{i}\chi^{(0)} = C_1(X)$$

we have

$$\left\langle x^0 \right\rangle = f_X(0) = 1$$

$$\left\langle x \right\rangle = \frac{1}{i} \frac{d f_X(k)}{dk}\Big|_{k=0} = \frac{1}{i} \chi f_X \Big|_{k=0} = C_1(X) \qquad (4.19)$$

so that $C_1(X)$ is just the mean of *X*.

$$\left\langle x^2 \right\rangle = \frac{1}{i^2} \frac{d^2 f_X(k)}{dk^2}\Big|_{k=0} = \frac{1}{i^2}\left(\chi' + \chi^2\right) f_X \Big|_{k=0}$$

$$= C_2(X) + C_1(X)^2$$

so that, with the help of (4.19), we have

$$C_2(X) = \left\langle x^2 \right\rangle - C_1(X)^2 = \left\langle x^2 \right\rangle - \left\langle x \right\rangle^2 \qquad (4.20)$$

which is just the variance of *X*. Simlarly

$$\left\langle x^3 \right\rangle = \frac{1}{i^3} \frac{d^3 f_X(k)}{dk^3}\Big|_{k=0} = \frac{1}{i^3}\left(\chi'' + 3\chi\chi' + \chi^3\right) f_X \Big|_{k=0}$$

$$= C_3(X) + 3C_2(X)C_1(X) + C_1(X)^3$$

so that

$$C_3(X) = \langle x^3 \rangle - 3C_2(X)C_1(X) - C_1(X)^3$$

$$= \langle x^3 \rangle - 3\left(\langle x^2 \rangle - \langle x \rangle^2\right)\langle x \rangle - \langle x \rangle^3$$

$$= \langle x^3 \rangle - 3\langle x^2 \rangle\langle x \rangle + 2\langle x \rangle^3 \qquad (4.21)$$

Finally,

$$\langle x^4 \rangle = \frac{1}{i^4}\frac{d^4 f_X(k)}{dk^4}\bigg|_{k=0} = \frac{1}{i^4}\left(\chi''' + 4\chi\chi'' + 3\chi'^2 + 6\chi^2\chi' + \chi^4\right) f_X\bigg|_{k=0}$$

$$= C_4(X) + 4C_1(X)C_3(X) + 3C_2(X)^2 + 6C_1(X)^2 C_2(X) + C_1(X)^4$$

so that

$$C_4(X) = \langle x^4 \rangle - 4\langle x \rangle\left(\langle x^3 \rangle - 3\langle x^2 \rangle\langle x \rangle + 2\langle x \rangle^3\right)$$

$$-3\left(\langle x^2 \rangle - \langle x \rangle^2\right)^2 - 6\langle x \rangle^2\left(\langle x^2 \rangle - \langle x \rangle^2\right) - \langle x \rangle^4$$

$$= \langle x^4 \rangle - 3\langle x^2 \rangle^2 - 4\langle x \rangle\langle x^3 \rangle + 12\langle x \rangle^2\langle x^2 \rangle - 6\langle x \rangle^4 \qquad (4.22)$$

Exercise 4.6

Consider the circular distribution

$$P_X(x) = \begin{cases} \dfrac{2}{\sqrt{\pi}}\sqrt{1-x^2} & \text{for } |x| \le 1 \\ 0 & |x| \ge 1 \end{cases}$$

Find $f_X(k)$ and the 1st four cumulants.

Answer

$$f_X(k) = \frac{2}{\sqrt{\pi}}\int_{-1}^{1} dx e^{ikx}\sqrt{1-x^2} = \frac{2}{k}J_1(k)$$

$$= \frac{2}{k}\left(\frac{k}{2} - \frac{k^3}{16} + \frac{k^5}{384} + \cdots\right)$$

$$= 1 - \frac{k^2}{8} + \frac{k^5}{192} + \cdots$$

so that from (4.15), we have

$$\langle x \rangle = \langle x^3 \rangle = 0$$

$$\langle x^2 \rangle = \frac{2!}{i^2}\left(-\frac{1}{8}\right) = \frac{1}{4} \qquad \langle x^4 \rangle = \frac{4!}{i^4}\left(\frac{1}{192}\right) = \frac{1}{8}$$

The cumulants are

$$C_1 = 0 \qquad C_2 = \frac{1}{4} \qquad C_3 = 0$$

$$C_4 = \frac{1}{8} - \frac{3}{16} = -\frac{1}{16}$$

4.D.4. Jointly Distributed Stochastic Variables

4.D.4a. Definitions

The stochastic variables $\{X_1, \cdots, X_n\}$ are **jointly distributed** if they are defined on the same sample space *S*. The **joint distribution function** is defined as

$$F_{X_1 \cdots X_n}(x_1, \cdots, x_n) \equiv \text{Proba}\left[\{X_1 < x_1\} \cap \cdots \cap \{X_n < x_n\}\right]$$

$$\equiv \text{Proba}\{X_1 < x_1, \cdots, X_n < x_n\} \qquad (4.23)$$

which means the **joint probability density** is given by

$$P_{X_1 \cdots X_n}(x_1, \cdots, x_n) = \frac{\partial^n F_{X_1 \cdots X_n}(x_1, \cdots, x_n)}{\partial x_1 \cdots \partial x_n} \qquad (4.24)$$

so that

$$F_{X_1 \cdots X_n}(x_1, \cdots, x_n) = \int_{-\infty}^{x_1} dx_1' \cdots \int_{-\infty}^{x_n} dx_n' P_{X_1 \cdots X_n}\left(x_1', \cdots, x_n'\right) \qquad (4.25)$$

4.D.4b. Two Stochastic Variables

Consider now the case of two stochastic variables X and Y.
By definition

$$F_{XY}\left(x,y\right)=\int_{-\infty}^{x}dx'\int_{-\infty}^{y}dy'P_{XY}\left(x',y'\right) \qquad \text{(a)}$$

with normalization

$$F_{XY}\left(\infty,\infty\right)=\int_{-\infty}^{\infty}dx'\int_{-\infty}^{\infty}dy'P_{XY}\left(x',y'\right)=1 \qquad (4.27)$$

and positivity

$$0\le P_{XY}\left(x,y\right)\le 1$$

It follows directly from eq(a) that

$$F_{XY}\left(-\infty,y\right)=F_{XY}\left(x,-\infty\right)=F_{XY}\left(-\infty,-\infty\right)=0$$

Also, for $x_2 > x_1$,

$$F_{XY}\left(x_2,y\right)-F_{XY}\left(x_1,y\right)=\left[\int_{-\infty}^{x_2}dx'-\int_{-\infty}^{x_1}dx'\right]\int_{-\infty}^{y}dy'P_{XY}\left(x',y'\right)$$

$$=\int_{x_1}^{x_2}dx'\int_{-\infty}^{y}dy'P_{XY}\left(x',y'\right) \quad =\text{Proba}\left\{x_1 < X \le x_2; Y\le y\right\} \ge 0 \qquad (4.26)$$

4.D.4c. The Reduced Distribution Function

The **reduced distribution function** $F_X(x)$ is defined by

$$F_X(x) = \int_{-\infty}^{x} dx' \int_{-\infty}^{\infty} dy' P_{XY}(x', y') = F_{XY}(x, \infty) \tag{4.28}$$

so that

$$P_X(x) = \frac{\partial F_X(x)}{\partial x} = \int_{-\infty}^{\infty} dy P_{XY}(x, y) \tag{4.29}$$

and similarly for *Y*.
The ***n*th moment** of *X* is defined as

$$\langle x^n \rangle = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy P_{XY}(x, y) x^n = \int_{-\infty}^{\infty} dx P_X(x) x^n \tag{4.30}$$

The joint moments of *X* and *Y* are

$$\langle x^n y^m \rangle = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy P_{XY}(x, y) x^n y^m \tag{4.31}$$

4.D.4d. Correlation Functions

The **covariance** of X and Y is defined by

$$Cov(X,Y) = \left\langle \left(x - \langle x\rangle\right)\left(y - \langle y\rangle\right)\right\rangle$$

$$= \langle xy\rangle - \langle x\rangle\langle y\rangle - \langle y\rangle\langle x\rangle + \langle x\rangle\langle y\rangle$$

$$= \langle xy\rangle - \langle x\rangle\langle y\rangle \qquad (4.32)$$

The **correlation** of X and Y is defined by

$$Cor(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} \qquad (4.33)$$

where the standard deviation is given by $\sigma_X = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$.

Properties

The correlation function is dimensionless with the following properties:

1. Symmetric: $Cor(X,Y) = Cor(Y,X)$.

2. $Cor(X,X) = 1, \qquad Cor(X,-X) = -1$.

3. $-1 \le Cor(X,Y) \le 1$.

4. $Cor(aX + b, cY + d) = Cor(X,Y)$ if $a, c \ne 0$.

Proofs

Properties (1-2) follow directly from definition (4.33).
To prove (3), we use the Schwarz inequality to get

$$\left|Cov(X,Y)\right|^2 = \left|\int\int dxdyP_{XY}\left(x - \langle x\rangle\right)\left(y - \langle y\rangle\right)\right|^2$$

$$\le \int\int dxdy\left|\sqrt{P_{XY}}\left(x - \langle x\rangle\right)\right|^2 \times \int\int dxdy\left|\sqrt{P_{XY}}\left(y - \langle y\rangle\right)\right|^2$$

$$= \int\int dxdyP_{XY}\left(x - \langle x\rangle\right)^2 \times \int\int dxdyP_{XY}\left(y - \langle y\rangle\right)^2$$

$$= \left\langle\left(x - \langle x\rangle\right)^2\right\rangle\left\langle\left(y - \langle y\rangle\right)^2\right\rangle \quad = \sigma_X{}^2\sigma_Y{}^2$$

Property (4) is the summary of the properties

$$Cor(X,a) = Cor(a,Y) = Cor(a,b) = 0$$

$$Cor(aX,Y) = Cor(X,aY) = Cor(X,Y)$$

each of which follows directly from definition (4.33).

Statistically Independent Variables

If *X* and *Y* are statistically independent, we have

1. $P_{XY}(x,y) = P_X(x)P_Y(y)$.

2. $\langle xy \rangle = \langle x \rangle \langle y \rangle$.

3. $\left\langle (x+y)^2 \right\rangle - \langle x+y \rangle^2 = \left\langle x^2 \right\rangle - \langle x \rangle^2 + \left\langle y^2 \right\rangle - \langle y \rangle^2$.

4. $Cor(X,Y) = 0$

Note that $Cor(X,Y) = 0$ does not imply *X* and *Y* are statistically independent.

Exercise 4.7

Show that $Cor(X,Y)$ is a measure of the dependence between *X* and *Y*.

Answer

Consider the special case $X = aY + b$.

A measure of the dependence between *X* and *Y* is the mean square error

$e = \left\langle (X - aY - b)^2 \right\rangle$. We are interested in the extrema of *e* as a function of *a* and *b*,

i.e.,

$$0 = \delta e = 2\left\langle (X - aY - b)(-Y\delta a - \delta b) \right\rangle$$

$$= 2\left[-\langle XY \rangle + a\left\langle Y^2 \right\rangle + b\langle Y \rangle \right]\delta a - 2\left[\langle X \rangle - a\langle Y \rangle - b \right]\delta b$$

Since δa and δb are arbitrary, we have

$$-\langle XY \rangle + a\left\langle Y^2 \right\rangle + b\langle Y \rangle = 0 \qquad (1)$$

$$\langle X \rangle - a\langle Y \rangle - b = 0 \qquad (2)$$

$$(1)+\langle Y\rangle(2)\Rightarrow \qquad a=\frac{\langle XY\rangle-\langle X\rangle\langle Y\rangle}{\langle Y^2\rangle-\langle Y\rangle^2} \quad =\frac{\sigma_X}{\sigma_Y}Cor(X,Y)$$

Thus, $Cor(X,Y)=0 \;\Rightarrow\; a=0$ so that X and Y are independent.

4.D.4e. Probability Density of Functions

The probability density $P_Z(z)$ for the stochastic variable $Z = G(X,Y)$ is given by

$$P_Z(z) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy P_{XY}(x,y)\,\delta\left[z - G(x,y)\right] \qquad (4.34)$$

Using

$$\delta(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ikx}$$

we have

$$P_Z(z) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy P_{XY}(x,y)\, e^{-ik\left[z - G(x,y)\right]}$$

which, on comparison with

$$P_Z(z) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ikz} f_Z(k)$$

gives

$$f_Z(k) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy P_{XY}(x,y)\, e^{ikG(x,y)} \qquad (4.35)$$

If X and Y are statistically independent, i.e.,

$$P_{XY}(x,y) = P_X(x) P_Y(y)$$

we have

$$f_Z(k) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy P_X(x) P_Y(y)\, e^{ikG(x,y)}$$

For example, if

$$G(x,y) = x + y$$

then

$$f_Z(k) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy P_X(x) P_Y(y)\, e^{ik(x+y)}$$

$$= f_X(k)\, f_Y(k)$$

i.e., the characteristic function of the sum of 2 independent stochastic variables is the product of their characteristic functions.

4.D.4f. Exercise 4.8

Let *X* and *Y* be statistically independent and Gaussian distributed with $\langle x \rangle = \langle y \rangle = 0$ and $\sigma_X = \sigma_Y = 1$. Find $P_{VW}(v,w)$, where $V = X + Y$ and $W = X - Y$. Are *V* and *W* independent?

Answer

Since *X* and *Y* are statistically independent, we have

$$P_{VW}(v,w) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy P_X(x) P_Y(y) \delta\left[v - v(x,y)\right] \delta\left[w - w(x,y)\right]$$

where

$$v(x,y) = x + y \qquad \text{and} \qquad w(x,y) = x - y \qquad \text{(a)}$$

Using

$$P_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) \qquad P_Y(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y^2\right)$$

we have

$$P_{VW}(v,w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \exp\left[-\frac{1}{2}\left(x^2 + y^2\right)\right] \delta\left[v - v(x,y)\right] \delta\left[w - w(x,y)\right]$$

To evaluate the delta functions is tantamount to finding $\delta\left[x - x(v,w)\right]$ and $\delta\left[y - y(v,w)\right]$, where $x(v,w)$ and $y(v,w)$ are the inverse functions of $v(x,y)$ and $w(x,y)$. Thus, we are dealing with the transformation $(v,w) \to (x,y)$.

Since delta functions are meaningful only inside integrals, care must also be taken to preserve the normalization

$$\int\int dxdy \delta\left[f(x,y)\right] = \int\int dvdw \delta\left[g(v,w)\right] = 1 \qquad \text{(b)}$$

for any arbitrary pairs of functions *f* and *g* that are related by the transformation.

Solving (a) for *x*, *y* gives

$$x(v,w) = \frac{1}{2}(v + w) \qquad \text{and} \qquad y(v,w) = \frac{1}{2}(v - w)$$

The jacobian for the transformation $(v,w)\to(x,y)$ is

$$J=\frac{\partial(x,y)}{\partial(v,w)}=\begin{vmatrix}\frac{\partial x}{\partial v} & \frac{\partial y}{\partial v}\\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w}\end{vmatrix}=\begin{vmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{vmatrix}=-\frac{1}{2}$$

so that, assuming the sign of the integrals are taken care of by a proper choice of integration limits, we have

$$dxdy=\left|\frac{\partial(x,y)}{\partial(v,w)}\right|dvdw=|J|dvdw=\frac{1}{2}dvdw$$

With the help of the identity (b), we see that

$$\left|\frac{\partial(x,y)}{\partial(v,w)}\right|\delta\left[f(x,y)\right]=\delta\left[g(v,w)\right]$$

Hence,

$$\frac{1}{2}\delta\left[x-\frac{1}{2}(v+w)\right]\delta\left[y-\frac{1}{2}(v-w)\right]=\delta\left[v-v(x,y)\right]\delta\left[w-w(x,y)\right]$$

and

$$P_{VW}(v,w)=\frac{1}{4\pi}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\exp\left[-\frac{1}{2}\left(x^2+y^2\right)\right]\delta\left[x-\frac{1}{2}(v+w)\right]\delta\left[y-\frac{1}{2}(v-w)\right]$$

$$=\frac{1}{4\pi}\exp\left[-\frac{1}{8}\left((v+w)^2+(v-w)^2\right)\right]$$

$$=\frac{1}{4\pi}\exp\left[-\frac{1}{4}\left(v^2+w^2\right)\right]$$

Since this is factorizable, V and W are statistically independent.

4.D.4g. Characteristic Functions

Generalizing (4.15) to the case of N stochastic variables, we define the characteristic function by

$$f_{X_1\cdots X_N}\left(k_1\cdots k_N\right)=\int_{-\infty}^{\infty}dx\cdots\int_{-\infty}^{\infty}dx_N P_{X_1\cdots X_N}\left(x_1\cdots x_N\right)\exp\left(ik_1x_1+\cdots+ik_Nx_N\right) \tag{4.36}$$

so that the joint moments for $n\le N$ are given by

$$\left\langle x_1\cdots x_n\right\rangle=\frac{1}{i^n}\frac{\partial}{\partial k_1}\cdots\frac{\partial}{\partial k_n}f_{X_1\cdots X_N}\bigg|_{k=0} \tag{4.37}$$

which $k=0$ is the shorthand for $k_1=\cdots=k_N=0$.

4.D.4h. Exercise 4.9

Consider the multivariant Gaussian distribution with zero means,

$$P_{X_1\cdots X_N}(x_1\cdots x_N)=\sqrt{\frac{\det\mathbf{G}}{(2\pi)^N}}\exp\left(-\frac{1}{2}\mathbf{x}^T\mathbf{G}\mathbf{x}\right)$$

where **G** is a symmetric $N\times N$ matrix, $\mathbf{x}^T=(x_1,\cdots,x_N)$ a row vector and

$$\mathbf{x}^T\mathbf{G}\mathbf{x}=\sum_{i=1}^{N}\sum_{j=1}^{N}x_i g_{ij} x_j$$

(a) Show that $P_{X_1\cdots X_N}(x_1\cdots x_N)$ is normalized to 1.

(b) Find $f_{X_1\cdots X_N}(k_1\cdots k_N)$.

(c) Show that $\langle x_i\rangle=0$ and all higher moments can be expressed as products of the 2nd moments $\langle x_i x_j\rangle$ and $\langle x_i^2\rangle$.

Answers

(a) (b) (c) (d)

4.D.4ha. Answer (a)

Since **G** is symmetric, there exists an orthogonal matrix **O**, with $\mathbf{O}^{-1} = \mathbf{O}^T$ **,** such that

$$\mathbf{OGO}^T = \quad = Diag\left(\gamma_1 \cdots \gamma_N\right)$$

$$\det \mathbf{G} = \det \mathbf{O}^T \times \det \quad \times \det \quad = \det \quad = \prod_{i=1}^{N} \gamma_i$$

Moreover, **G** is positive definite so that $\gamma_j > 0 \quad \forall\, j$.

Let $\mathbf{y} = \mathbf{Ox}$ or $y_i = \sum_{j=1}^{N} o_{ij} x_j$. Since **O** is independent of **x**, we have

$$\mathbf{x}^T \mathbf{Gx} = \mathbf{y}^T \mathbf{OGO}^T \mathbf{y} = \mathbf{y}^T \quad = \sum_{i=1}^{N} \gamma_i y_i^2$$

$$\frac{\partial y_i}{\partial x_j} = o_{ij} \qquad \Rightarrow \qquad \frac{\partial\left(y_1 \cdots y_N\right)}{\partial\left(x_1 \cdots x_N\right)} = \det\left(\frac{\partial y_i}{\partial x_j}\right) = \det \mathbf{O} = 1$$

so that

$$\int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_N P_{X_1 \cdots X_N}\left(x_1 \cdots x_N\right) = \sqrt{\frac{\det\left(\mathbf{G}\right)}{\left(2\pi\right)^N}} \int_{-\infty}^{\infty} dy_1 \cdots \int_{-\infty}^{\infty} dy_N \exp\left(-\frac{1}{2}\mathbf{y}^T \quad \right)$$

$$= \sqrt{\frac{1}{\left(2\pi\right)^N} \prod_{j=1}^{N} \gamma_j} \int_{-\infty}^{\infty} dy_1 \cdots \int_{-\infty}^{\infty} dy_N \exp\left(-\frac{1}{2}\sum_{i=1}^{N} \gamma_i y_i^2\right)$$

$$= \prod_{i=1}^{N}\left[\sqrt{\frac{\gamma_i}{2\pi}} \int_{-\infty}^{\infty} dy_i \exp\left(-\frac{1}{2}\gamma_i y_i^2\right)\right]$$

$$= \prod_{i=1}^{N} 1 \quad = 1$$

4.D.4hb. Answer (b)

The characteristic function is

$$f_{X_1\cdots X_N}\left(k_1\cdots k_N\right)=\sqrt{\frac{\det\mathbf{G}}{\left(2\pi\right)^N}}\int_{-\infty}^{\infty}dx_1\cdots\int_{-\infty}^{\infty}dx_N\,\exp\left(i\mathbf{k}^T\mathbf{x}-\frac{1}{2}\mathbf{x}^T\mathbf{G}\mathbf{x}\right)$$

Now,

$$i\mathbf{k}^T\mathbf{x}-\frac{1}{2}\mathbf{x}^T\mathbf{G}\mathbf{x}=i\left(\mathbf{O}\mathbf{k}\right)^T\mathbf{y}-\frac{1}{2}\mathbf{y}^T$$

$$=\sum_{j=1}^{N}\left(im_jy_j-\frac{1}{2}\gamma_jy_j^2\right)\qquad\text{where}\qquad\mathbf{m}=\mathbf{O}\mathbf{k}$$

$$=\sum_{j=1}^{N}\left[-\frac{1}{2}\gamma_j\left(y_j-i\frac{m_j}{\gamma_j}\right)^2-\frac{m_j^2}{2\gamma_j}\right]$$

$$=-\frac{1}{2}\mathbf{z}^T\quad-\frac{1}{2}\quad{}^T\quad{}^{-1}$$

where $z_j=y_j-i\dfrac{m_j}{\gamma_j}$ and $\mathbf{\Gamma}^{-1}=Diag\left(\gamma_1^{-1},\cdots,\gamma_N^{-1}\right)$.

Hence

$$f_{X_1\cdots X_N}\left(k_1\cdots k_N\right)=\exp\left(-\frac{1}{2}\mathbf{m}^T\quad{}^{-1}\quad\right)\sqrt{\frac{\det\mathbf{G}}{\left(2\pi\right)^N}}\int_{-\infty}^{\infty}dz_1\cdots\int_{-\infty}^{\infty}dz_N\,\exp\left(-\frac{1}{2}\quad{}^T\quad\right)$$

$$=\exp\left(-\frac{1}{2}\mathbf{m}^T\quad{}^{-1}\quad\right)$$

$$=\exp\left(-\frac{1}{2}\mathbf{k}^T\mathbf{G}^{-1}\mathbf{k}\right)$$

where

$$\mathbf{m}^T\quad{}^{-1}\quad=\quad{}^T\quad{}^T\quad{}^{-1}\quad=\quad{}^T\quad{}^{-1}$$

4.D.4hc. Answer (c)

Since

$$f_{X_1\cdots X_N}\left(k_1\cdots k_N\right)=\exp\left(-\frac{1}{2}\mathbf{k}^T\mathbf{G}^{-1}\mathbf{k}\right)$$

with $\quad f_{X_1\cdots X_N}\Big|_{k=0}=1$.

All moments can be calculated using

$$\frac{\partial}{\partial k_j}\left(\mathbf{k}^T\mathbf{G}^{-1}\mathbf{k}\right)=\frac{\partial}{\partial k_j}\sum_{mn}k_m g_{mn}^{-1}k_n=\sum_n g_{jn}^{-1}k_n+\sum_m k_m g_{mj}^{-1}$$

$$=2\sum_n g_{jn}^{-1}k_n \qquad \text{[since } \mathbf{G} \text{ is symmetric]}$$

$$=2\left(\mathbf{G}^{-1}\mathbf{k}\right)_j$$

$$\frac{\partial^2}{\partial k_i\partial k_j}\left(\mathbf{k}^T\mathbf{G}^{-1}\mathbf{k}\right)=2\frac{\partial}{\partial k_i}\left(\mathbf{G}^{-1}\mathbf{k}\right)_j=2\frac{\partial}{\partial k_i}\sum_n g_{jn}^{-1}k_n=2g_{ji}^{-1}=2g_{ij}^{-1}$$

1st Moment

Since

$$\frac{\partial f_{X_1\cdots X_N}}{\partial k_j}=f_{X_1\cdots X_N}\frac{\partial}{\partial k_j}\left(-\frac{1}{2}\mathbf{k}^T\mathbf{G}^{-1}\mathbf{k}\right)$$

$$=-\left(\mathbf{G}^{-1}\mathbf{k}\right)_j f_{X_1\cdots X_N}$$

we have

$$\left\langle x_j\right\rangle=\frac{1}{i}\frac{\partial f_{X_1\cdots X_N}}{\partial k_j}\Bigg|_{k=0}=-\frac{1}{2i}\cdot\frac{\partial}{\partial k_j}\left(\mathbf{k}^T\mathbf{G}^{-1}\mathbf{k}\right)\Bigg|_{k=0}=0$$

2nd Moment

$$\frac{\partial^2 f_{X_1\cdots X_N}}{\partial k_i\partial k_j}=\frac{\partial}{\partial k_i}\left[-\left(\mathbf{G}^{-1}\mathbf{k}\right)_j f_{X_1\cdots X_N}\right]$$

$$=\left[-g_{ij}^{-1}+\left(\mathbf{G}^{-1}\mathbf{k}\right)_i\left(\mathbf{G}^{-1}\mathbf{k}\right)_j\right]f_{X_1\cdots X_N}$$

so that

$$\left\langle x_i x_j \right\rangle = \frac{1}{i^2} \frac{\partial^2 f_{X_1 \cdots X_N}}{\partial k_i \partial k_j} \Bigg|_{k=0} = g_{ij}^{-1}$$

3rd Moment

$$\frac{\partial^3 f_{X_1 \cdots X_N}}{\partial k_i \partial k_j \partial k_l} = \frac{\partial}{\partial k_i} \left\{ \left[-g_{jl}^{-1} + \left(\mathbf{G}^{-1}\mathbf{k}\right)_j \left(\mathbf{G}^{-1}\mathbf{k}\right)_l \right] f_{X_1 \cdots X_N} \right\}$$

$$= \left\{ g_{ij}^{-1} \left(\mathbf{G}^{-1}\mathbf{k}\right)_l + g_{il}^{-1} \left(\mathbf{G}^{-1}\mathbf{k}\right)_j - \left(\mathbf{G}^{-1}\mathbf{k}\right)_i \left[-g_{jl}^{-1} + \left(\mathbf{G}^{-1}\mathbf{k}\right)_j \left(\mathbf{G}^{-1}\mathbf{k}\right)_l \right] \right\} f_{X_1 \cdots X_N}$$

so that

$$\left\langle x_i x_j x_l \right\rangle = \frac{1}{i^3} \frac{\partial^3 f_{X_1 \cdots X_N}}{\partial k_i \partial k_j \partial k_l} \Bigg|_{k=0} = 0$$

4th Moment

$$\frac{\partial^4 f_{X_1 \cdots X_N}}{\partial k_i \partial k_j \partial k_l \partial k_m} = \frac{\partial}{\partial k_i} \left\{ \left\{ g_{jl}^{-1} \left(\mathbf{G}^{-1}\mathbf{k}\right)_m + g_{jm}^{-1} \left(\mathbf{G}^{-1}\mathbf{k}\right)_l \right. \right.$$

$$\left. \left. - \left(\mathbf{G}^{-1}\mathbf{k}\right)_j \left[-g_{lm}^{-1} + \left(\mathbf{G}^{-1}\mathbf{k}\right)_l \left(\mathbf{G}^{-1}\mathbf{k}\right)_m \right] \right\} f_{X_1 \cdots X_N} \right\}$$

$$= \left\{ g_{jl}^{-1} g_{im}^{-1} + g_{jm}^{-1} g_{il}^{-1} - g_{ij}^{-1} \left[-g_{lm}^{-1} + \left(\mathbf{G}^{-1}\mathbf{k}\right)_l \left(\mathbf{G}^{-1}\mathbf{k}\right)_m \right] \right.$$

$$+ \left(\mathbf{G}^{-1}\mathbf{k}\right)_i \left\{ g_{jl}^{-1} \left(\mathbf{G}^{-1}\mathbf{k}\right)_m + g_{jm}^{-1} \left(\mathbf{G}^{-1}\mathbf{k}\right)_l \right.$$

$$\left. \left. - \left(\mathbf{G}^{-1}\mathbf{k}\right)_j \left[-g_{lm}^{-1} + \left(\mathbf{G}^{-1}\mathbf{k}\right)_l \left(\mathbf{G}^{-1}\mathbf{k}\right)_m \right] \right\} \right\} f_{X_1 \cdots X_N}$$

so that

$$\left\langle x_i x_j x_l x_m \right\rangle = \frac{1}{i^4} \frac{\partial^4 f_{X_1 \cdots X_N}}{\partial k_i \partial k_j \partial k_l \partial k_m} \Bigg|_{k=0}$$

$$= g_{jl}^{-1} g_{im}^{-1} + g_{jm}^{-1} g_{il}^{-1} + g_{ij}^{-1} g_{lm}^{-1}$$

$$= \left\langle x_j x_l \right\rangle \left\langle x_i x_m \right\rangle + \left\langle x_j x_m \right\rangle \left\langle x_i x_l \right\rangle + \left\langle x_i x_j \right\rangle \left\langle x_l x_m \right\rangle$$

4.D.4hd. Wick's Theorem

It can be shown by induction that

$$\langle x_1 \cdots x_{2n+1} \rangle = 0$$

$$\langle x_1 \cdots x_{2n} \rangle = \sum [\text{All possible pairwise averages}]$$

which is the simplest form of the Wick's theorem.

The number of terms in the sum is $\frac{(2n)!}{2^n n!}$.

Proof

The number of ways to put 2*n* objects in 2*n* places is (2*n*)!.

Consider these as *n* pairs of objects.

The number of permutations among these pairs is *n*!.

The number of permutations within each pair is 2^n.

Hence, the number of ways to pick *n* pairs out of 2*n* objects irregardless of order is $\frac{(2n)!}{2^n n!}$.

4.E. Binomial Distributions

Consider N experiments, each of which has only 2 possible outcome.
The probability distribution for one of the outcomes is called the **binomial distribution**.

In the limit of very large N, the binomial ditribution can be approximated by

1. the **Gaussian** (**normal**) **distribution** if the probabilities of the 2 outcomes are not too disparate.
2. the **Poisson distribution** if the outcome of interest has a diminishing probability.

4.E.1. Binomial Distribution

Consider a sequence of N statistically independent trials, each of which has only 2 possible outcomes: failure (0) and success (1) of probabilities q and p, respectively. The respective numbers of occurrences are m and n. By definition, $q+p=1$ and $m+n=N$. Statistical independence of the trials means that the probability for a given permutation of such a result is $p^n q^m = p^n q^{N-n}$. Since the number of such permutations is $\frac{N!}{n!m!}$, the probability for any combination of such a result is

$$P_N(n) = \frac{N!}{n!m!} p^n q^m = \frac{N!}{n!(N-n)!} p^n q^{N-n} \qquad (4.38)$$

which is called the **binomial distribution**.

Normalization

According to the binomial theorem,

$$\begin{aligned}\sum_{n=0}^{N} P_N(n) &= \sum_{n=0}^{N} \frac{N!}{n!m!} p^n q^m \\ &= \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} p^n q^{N-n} \\ &= (p+q)^N = 1 \qquad (4.39)\end{aligned}$$

so that the normalization condition is satisfied.

Stochastic Variables

The foregoing can be expressed in terms of stochastic variables as follows.
Let X_i be the stochastic variable for the ith trial. We have

$$P_{X_i}(x) = q\delta(x) + p\delta(x-1)$$

$$f_{X_i}(k) = \int_{-\infty}^{\infty} dx e^{ikx} \left[q\delta(x) + p\delta(x-1) \right] = q + pe^{ik} \qquad (4.39a)$$

Exercise 4.10

The probability of an archer hitting the target is 1/3. What is the probability of his hitting the target at least 3 times in 5 attempts.

Answer

$$P_5(3) = \frac{5!}{3!2!}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2 = \frac{5\cdot 2\cdot 4}{3^5} = \frac{40}{243}$$

$$P_5(4) = \frac{5!}{4!}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right) = \frac{5\cdot 2}{3^5} = \frac{10}{243}$$

$$P_5(5) = \frac{5!}{5!}\left(\frac{1}{3}\right)^5 = \frac{1}{3^5} = \frac{1}{243}$$

$$\Rightarrow \qquad P = P_5(3) + P_5(4) + P_5(5) = \frac{51}{243} \simeq 0.21$$

Additive Outcomes

Consider the additive outcome of the stochastic variables X_i after N trials

$$Y_N = \sum_{i=1}^{N} X_i$$

Since X_i are statistically independent, we have, according to (4.34-5),

$$P_{Y_N}(y) = \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_N P_{X_1}(x_1)\cdots P_{X_N}(x_N)\,\delta\left(y - \sum_{i=1}^{N} x_i\right) \tag{4.40}$$

and

$$f_{Y_N}(k) = \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_N P_{X_1}(x_1)\cdots P_{X_N}(x_N)\exp\left(ik\sum_{i=1}^{N} x_i\right)$$

$$= \prod_{i=1}^{N}\left[\int_{-\infty}^{\infty} dx_i P_{X_i}(x_i)\exp(ikx_i)\right]$$

$$= \prod_{i=1}^{N} f_{X_i}(k) = \left(q + pe^{ik}\right)^N \tag{4.41}$$

where (4.39a) was used in the last step. Using the binomial theorem, we get

$$f_{Y_N}(k) = \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} q^{N-n} p^n e^{ikn} \tag{4.42}$$

Taking the inverse Fourier transform gives

$$P_{Y_N}(y) = \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} q^{N-n} p^n \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(n-y)}$$

$$= \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} q^{N-n} p^n \delta(y-n)$$

$$= \sum_{n=0}^{N} P_N(n) \delta(y-n) \qquad (4.43)$$

where $P_N(n)$ is the probability of n successes in N trials [see (4.38)]. Note that

$$\langle y^\alpha \rangle = \int_{-\infty}^{\infty} dy P_{Y_N}(y) y^\alpha = \sum_{n=0}^{N} P_N(n) \int_{-\infty}^{\infty} dy \delta(y-n) y^\alpha$$

$$= \sum_{n=0}^{N} P_N(n) n^\alpha \quad = \langle n^\alpha \rangle$$

1st Moment

The 1st moment, or mean, of outcome $+1$ is

$$\langle y \rangle = \langle n \rangle = \sum_{n=0}^{N} n P_N(n) = \sum_{n=0}^{N} \frac{nN!}{n!(N-n)!} p^n q^{N-n}$$

$$= p \frac{\partial}{\partial p} \sum_{n=0}^{N} P_N(n) \quad = p \frac{\partial}{\partial p} (p+q)^N$$

$$= Np(p+q)^{N-1} = Np \qquad (4.44)$$

Alternatively, we can make use of (4.41) to get

$$\langle y \rangle = \frac{1}{i} \frac{\partial f_{Y_N}}{\partial k} \bigg|_{k=0} \quad = \frac{1}{i} N \left(q + pe^{ik}\right)^{N-1} pie^{ik} \bigg|_{k=0}$$

$$= N(q+p)^{N-1} p \quad = Np$$

2nd Moment

Similarly, the 2nd moment is

$$\langle y^2 \rangle = \langle n^2 \rangle = \sum_{n=0}^{N} n^2 P_N(n) = \sum_{n=0}^{N} \frac{n^2 N!}{n!(N-n)!} p^n q^{N-n}$$

$$= \left(p \frac{\partial}{\partial p}\right)^2 \sum_{n=0}^{N} P_N(n) \quad = p \frac{\partial}{\partial p} \left[Np(p+q)^{N-1} \right]$$

$$= Np\left[(p+q)^{N-1} + (N-1) p (p+q)^{N-2} \right]$$

$$= Np\left[1 + (N-1)p\right] \quad = Np(Np+q)$$

$$= (Np)^2 + Npq \qquad (4.46)$$

Alternatively

$$\langle y^2 \rangle = \frac{1}{i^2} \frac{\partial^2 f_{Y_N}}{\partial k^2} \bigg|_{k=0}$$

$$= \frac{1}{i^2} \left[N(N-1)\left(q + pe^{ik}\right)^{N-2} \left(pie^{ik}\right)^2 + N\left(q + pe^{ik}\right)^{N-1} pi^2 e^{ik} \right] \bigg|_{k=0}$$

$$= N(N-1)(q+p)^{N-2} p^2 + N(q+p)^{N-1} p$$

$$= N(N-1)p^2 + Np \quad = Np\left[(N-1)p + 1\right] \quad = Np(Np + q)$$

The variance is therefore

$$\sigma_N^2 = \langle n^2 \rangle - \langle n \rangle^2 = Npq \qquad (4.47a)$$

so that the standard deviation is

$$\sigma_N = \sqrt{Npq} \qquad (4.47)$$

Finally, the fractional deviation is

$$\frac{\sigma_N}{\langle n \rangle} = \sqrt{\frac{q}{Np}} \qquad (4.48)$$

which goes to zero as $N \to \infty$. Now, a small σ_N means *n* is distributed close to the mean $\langle n \rangle = Np$. Hence, as $N \to \infty$, we have $n \simeq Np$ or $\frac{n}{N} \simeq p$. The binomial distribution for $N = 10$ and $p = \frac{1}{3}$ is shown in Fig.4.2.

4.E.2. Gaussian Distribution

Consider the stochastic variable

$$Z_N = \frac{Y_N - \langle y \rangle}{\sigma_Y} = \frac{Y_N - Np}{\sqrt{Npq}}$$

$$\Rightarrow \qquad \langle z \rangle = \frac{\langle y \rangle - \langle y \rangle}{\sigma_Y} = 0$$

Using eq(4.34-5), we have

$$P_{Z_N}(z) = \int_{-\infty}^{\infty} dy P_{Y_N}(y) \delta\left[z - \frac{y - \langle y \rangle}{\sigma_Y} \right]$$

$$= \int_{-\infty}^{\infty} dy P_{Y_N}(y) \delta\left(z - \frac{y - Np}{\sqrt{Npq}} \right) \qquad (4.49)$$

$$f_{Z_N}(k) = \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dy P_{Y_N}(y) \delta\left(z - \frac{y - Np}{\sqrt{Npq}} \right) e^{ikz}$$

$$= \int_{-\infty}^{\infty} dy P_{Y_N}(y) \exp\left(ik \frac{y - Np}{\sqrt{Npq}} \right)$$

$$= \exp\left(-ik\sqrt{\frac{Np}{q}} \right) \int_{-\infty}^{\infty} dy P_{Y_N}(y) \exp\left(i \frac{k}{\sqrt{Npq}} y \right)$$

$$= \exp\left(-ik\sqrt{\frac{Np}{q}} \right) f_{Y_N}\left(\frac{k}{\sqrt{Npq}} \right)$$

$$= \exp\left(-ik\sqrt{\frac{Np}{q}} \right) \left[q + p \exp\left(\frac{ik}{\sqrt{Npq}} \right) \right]^N \qquad \text{[eq(4.41) used]}$$

$$= \left[q \exp\left(-ik\sqrt{\frac{p}{Nq}} \right) + p \exp\left(\frac{ik}{\sqrt{Npq}} (1 - p) \right) \right]^N$$

$$= \left[q \exp\left(-ik\sqrt{\frac{p}{Nq}} \right) + p \exp\left(ik\sqrt{\frac{q}{Np}} \right) \right]^N \qquad (4.50)$$

$N \to \infty$

In the limit $N \to \infty$, we have

$$q \exp\left(-ik\sqrt{\frac{p}{Nq}} \right) \simeq q - ik\sqrt{\frac{pq}{N}} - k^2 \frac{p}{2N} + \sum_{j=3}^{\infty} \frac{(-ik)^j}{j!} \left(\frac{p}{Nq} \right)^{j/2} q$$

$$p \exp\left(ik\sqrt{\frac{q}{Np}} \right) \simeq p + ik\sqrt{\frac{pq}{N}} - k^2 \frac{q}{2N} + \sum_{j=3}^{\infty} \frac{(ik)^j}{j!} \left(\frac{q}{Np} \right)^{j/2} p$$

so that (4.50) becomes

$$f_{Z_N}(k) = \left[1 - \frac{k^2}{2N}(1 + R_N) \right]^N \qquad (4.51)$$

where

$$R_N = -\frac{2N}{k^2} \sum_{j=3}^{\infty} \left[\frac{(-ik)^j}{j!} \left(\frac{p}{Nq} \right)^{j/2} q + \frac{(ik)^j}{j!} \left(\frac{q}{Np} \right)^{j/2} p \right]$$

$$= 2\sum_{j=3}^{\infty} \frac{1}{j!} \left(\frac{ik}{\sqrt{N}} \right)^{j-2} \left[(-)^j \left(\frac{p}{q} \right)^{j/2} q + \left(\frac{q}{p} \right)^{j/2} p \right]$$

$$= 2\sum_{j=3}^{\infty} \frac{1}{j!} \left(\frac{ik}{\sqrt{N}} \right)^{j-2} \left[\frac{(-)^j p^j q + q^j p}{q^{j/2} p^{j/2}} \right] \qquad (4.52)$$

Thus, $R_N \to 0$ as $N \to \infty$ so that (4.51) becomes

$$f_Z(k) = \lim_{N\to\infty} f_{Z_N}(k) = \lim_{N\to\infty} \left(1 - \frac{k^2}{2N} \right)^N = \exp\left(-\frac{k^2}{2} \right) \qquad (4.53)$$

where we've used

$$\lim_{N\to\infty} \left(1 + \frac{x}{N} \right)^N = e^x \qquad (4.54)$$

Proof of (4.54) is as follows. Let

$$f = \left(1 + \frac{x}{N} \right)^N$$

we have

$$\ln f = N \ln\left(1 + \frac{x}{N} \right) = N \sum_{n=1}^{\infty} \left(\frac{x}{N} \right)^n (-)^{n+1} \to x \quad \text{as} \quad N \to \infty. \quad \text{QED}$$

Now

$$P_Z(z) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp\left(-ikz - \frac{1}{2}k^2\right)$$

$$= \exp\left(-\frac{z^2}{2}\right) \int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp\left[-\frac{1}{2}(k+iz)^2\right]$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) \qquad (4.55)$$

Hence,

$$\langle z \rangle = \int_{-\infty}^{\infty} dz P_Z(z) z = 0 \qquad \text{[since } P_Z \text{ is even]}$$

$$\langle z^2 \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz \exp\left(-\frac{z^2}{2}\right) z = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\pi}}{2} 2^{3/2} = 1$$

where we've used

$$\int_{0}^{\infty} dx \exp\left(-ax^2\right) x^n = \frac{1}{2} \Gamma\left(\frac{n+1}{2}\right) a^{-(n+1)/2}$$

Gaussian Probability Density

From $z = \dfrac{y - \langle y \rangle}{\sigma_Y}$ we have $y = \sigma_Y z + \langle y \rangle$ so that

$$P_{Y_N}(y) = \int_{-\infty}^{\infty} dz P_{Z_N}(z) \delta\left[y - \left(\langle y \rangle + \sigma_Y z\right)\right]$$

For large N, $P_{Z_N}(z) \simeq P_Z(z)$ so that with (4.55), we have

$$P_{Y_N}(y) = \int_{-\infty}^{\infty} dz \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) \delta\left[y - \left(\langle y \rangle + \sigma_Y z\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}\sigma_Y} \int_{-\infty}^{\infty} dz \exp\left(-\frac{z^2}{2}\right) \delta\left[z - \frac{y - \langle y \rangle}{\sigma_Y}\right]$$

$$= \frac{1}{\sqrt{2\pi}\sigma_Y} \exp\left[-\frac{1}{2\sigma_Y^2}\left(y - \langle y \rangle\right)^2\right] \qquad (4.56)$$

which is the **Gaussian probability density** for the number of successes after a large number of trials. Note that it is given entirely in terms of the 1st two moments, $\langle y \rangle$ and $\langle y^2 \rangle$.

4.E.3. Poisson Distribution

In the limit $N \to \infty$, $p \to 0$ such that $\langle n \rangle = Np = a \ll N$, the binomial distribution becomes the **Poisson distribution**. Substituting $p = \frac{\langle n \rangle}{N} = \frac{a}{N}$ into

$$f_{Y_N}(k) = \left(q + pe^{ik}\right)^N \tag{4.41}$$

we have

$$f_{Y_N}(k) = \left(1 - \frac{a}{N} + \frac{a}{N} e^{ik}\right)^N \tag{4.57}$$

In the limit $N \to \infty$, we have

$$f_Y(k) = \lim_{N\to\infty} f_{Y_N}(k) \quad = \exp\left(-a + ae^{ik}\right)$$

$$= e^{-a} \exp\left(ae^{ik}\right) \quad = e^{-a} \sum_{n=0}^{\infty} \frac{a^n}{n!} e^{ink} \tag{4.58}$$

Taking the inverse Fourier transform gives

$$P_Y(y) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-iky} f_Y(k) \quad = e^{-a} \sum_{n=0}^{\infty} \frac{a^n}{n!} \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ik(y-n)}$$

$$= e^{-a} \sum_{n=0}^{\infty} \frac{a^n}{n!} \delta(y-n) \tag{4.59}$$

$$= \sum_{n=0}^{\infty} P(n) \delta(y-n)$$

where [see (4.43)],

$$P(n) = \frac{a^n}{n!} e^{-a} \tag{4.60}$$

is commonly called the **Poisson distribution** though it is actually a probability density. Note that it is given solely by the 1st moment $\langle n \rangle$.

Normalization

$$\sum_{n=0}^{\infty} P(n) = e^{-a} \sum_{n=0}^{\infty} \frac{a^n}{n!} = e^{-a} e^{a} = 1$$

1st Moment

$$\langle n \rangle = \sum_{n=0}^{\infty} P(n) n = e^{-a} a \frac{\partial}{\partial a} \sum_{n=0}^{\infty} \frac{a^n}{n!} = e^{-a} a \frac{\partial}{\partial a} e^a = a$$

Exercise 4.11

Consider the neutron bombardment of a thin sheet of gold foil.
Assuming the average number of hits is 2, find

(a) The probability of no hit.

(b) The probability of exactly 2 hit.

Answer

(a) $P(0) = e^{-2} \frac{2^0}{0!} = e^{-2} \simeq 0.135$.

(b) $P(2) = e^{-2} \frac{2^2}{2!} = e^{-2} \cdot 2 \simeq 0.27$

4.E.4. Binomial Random Walk

Consider a particle constrainted to move along the *x*-axis by steps of equal lengths Δ but random directions taken at fixed time intervals τ. Let the probability of its moving to the right and left be *p* and $q = 1 - p$, respectively.

The movement at step *i* can therefore be decribed by a stochastic variable X_i with realization $x = \pm\Delta$, probability density

$$P_{X_i}(x) = p\delta(x-\Delta) + q\delta(x+\Delta)$$

and characteristic function

$$\begin{aligned} f_{X_i}(k) &= \int_{-\infty}^{\infty} dx e^{ikx}\left[p\delta(x-\Delta)+q\delta(x+\Delta)\right] \\ &= pe^{ik\Delta} + qe^{-ik\Delta} \\ &= \cos k\Delta \qquad \text{if} \qquad p = q = \frac{1}{2} \end{aligned}$$

The net displacement after *N* step is given by $Y_N = \sum_{i=1}^{N} X_i$. The characteristic function, according to (4.40-1), is

$$\begin{aligned} f_{Y_N}(k) &= \prod_{i=1}^{N} f_{X_i}(k) \quad = (\cos k\Delta)^N \\ &= \left(1 - \frac{1}{2}k^2\Delta^2 + \frac{1}{4!}k^4\Delta^4 - \cdots\right)^N \\ &\simeq 1 - \frac{1}{2}Nk^2\Delta^2 + \left[\frac{N(N-1)}{8} + \frac{N}{4!}\right]k^4\Delta^4 - \cdots \qquad (4.61) \end{aligned}$$

Hence,

$$\langle y \rangle = \frac{1}{i}\frac{\partial f_{Y_N}}{\partial k}\bigg|_{k=0} \quad = \frac{1}{i}\left(-Nk\Delta^2 + 4\left[\frac{N(N-1)}{8} + \frac{N}{4!}\right]k^3\Delta^4 + \cdots\right)\bigg|_{k=0} = 0$$

$$\langle y^2 \rangle = \frac{1}{i^2}\frac{\partial^2 f_{Y_N}}{\partial k^2}\bigg|_{k=0} \quad = \frac{1}{i^2}\left(-N\Delta^2 + 12\left[\frac{N(N-1)}{8} + \frac{N}{4!}\right]k^2\Delta^4 + \cdots\right)\bigg|_{k=0} = N\Delta^2$$

$$\sigma_{Y_N} = \sqrt{\langle y^2 \rangle - \langle y \rangle^2} = \Delta\sqrt{N}$$

For a given *N*, there are 2^N possible realizations. Three of them for the case

$N = 2000$ are shown in Fig.4.4.

Differential Equation

Consider the limits $\Delta, \tau \to 0$ with $N \to \infty$.

Setting $f_Y(k, N\tau) \equiv f_{Y_N}(k)$, we have

$$f_Y\left[k,(N+1)\tau\right] - f_Y(k,N\tau) = (\cos k\Delta)^{N+1} - (\cos k\Delta)^N$$

$$= (\cos k\Delta - 1)(\cos k\Delta)^N$$

$$= \left(-\frac{1}{2}k^2\Delta^2 + \frac{1}{4!}k^4\Delta^4 - \cdots\right) f_Y(k, N\tau) \qquad (4.62)$$

Hence, with $N\tau = t$, we have

$$\frac{\partial f_Y(k,t)}{\partial t} = \lim_{\tau\to 0} \frac{f_Y(k,t+\tau) - f_Y(k,t)}{\tau}$$

$$= \lim_{\tau\to 0} \frac{f_Y\left[k,(N+1)\tau\right] - f_Y(k,N\tau)}{\tau} \qquad (4.63)$$

$$= \lim_{\tau\to 0} \left(-\frac{1}{2}k^2\frac{\Delta^2}{\tau} + \frac{1}{4!}k^4\frac{\Delta^4}{\tau} - \cdots\right) f_Y(k,t)$$

$$= -Dk^2 f_Y(k,t) \qquad \text{as} \quad \Delta \to 0 \qquad (4.64)$$

where the diffusion coefficient D is defined as

$$D = \lim_{\Delta,\tau\to 0} \frac{\Delta^2}{2\tau}$$

For a fixed k, the general solution to (4.64) is

$$f_Y(k,t) = f_Y(k,0)\exp\left(-Dk^2 t\right)$$

$$= \exp\left(-Dk^2 t\right) \qquad \text{if} \qquad f_Y(k,0) = 1 \qquad (4.66)$$

Taking the inverse Fourier transform gives the probability distribution

$$P(y,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp\left(-iky - Dk^2 t\right)$$

$$= \exp\left(-\frac{y^2}{4Dt}\right) \int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp\left[-Dt\left(k + \frac{iy}{2Dt}\right)^2\right]$$

$$= \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{y^2}{4Dt} \right) \qquad (4.67)$$

which is a Gaussian with $\sigma = \sqrt{2Dt}$.

Alternative Derivation

Consider the stochastic variable $Z_N = \frac{Y_N}{\Delta\sqrt{N}}$. The associated characteristic function , according to (4.40-1), is

$$f_{Z_N}(k) = \prod_{j=1}^{N} \left[\int_{-\infty}^{\infty} dx_j \exp\left(\frac{ik}{\Delta\sqrt{N}} x_j \right) P_{X_j}(x_j) \right]$$

$$= \prod_{j=1}^{N} f_{X_j}\left(\frac{k}{\Delta\sqrt{N}} \right) = \left(\cos\frac{k}{\sqrt{N}} \right)^N \qquad (4.68)$$

$$\simeq \left(1 - \frac{k^2}{2N} + \frac{k^4}{4!\,N^2} - \cdots \right)^N$$

Hence

$$f_Z(k) = \lim_{N\to\infty} f_{Z_N}(k) = \exp\left(-\frac{k^2}{2} \right) \qquad (4.69)$$

$$P_Z(z) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp\left(-ikz - \frac{1}{2}k^2 \right)$$

$$= \exp\left(-\frac{z^2}{2} \right) \int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp\left(-\frac{1}{2}(k+iz)^2 \right)$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2} \right)$$

For very large *N*, we have $P_{Z_N}(z) \simeq P_Z(z)$ so that from (4.49), we have

$$P_{Y_N}(y) \simeq \int_{-\infty}^{\infty} dz\,\delta\left(y - \Delta\sqrt{N} z \right) P_Z(z)$$

$$= \frac{1}{\Delta\sqrt{N}} \int_{-\infty}^{\infty} dz\,\delta\left(z - \frac{y}{\Delta\sqrt{N}} \right) P_Z(z)$$

$$= \frac{1}{\sqrt{2\pi\Delta^2 N}} \exp\left(-\frac{y^2}{2\Delta^2 N}\right) \qquad (4.70)$$

If the partical takes *n* per unit time, we have $N = nt$. Setting $D = \frac{1}{2} n\Delta^2$, we have

$$P(y,t) = \lim_{N\to\infty} P_{Y_N}(y) = \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{y^2}{4Dt}\right) \qquad (4.71)$$

See Fig.4.5.

4.F. A Central Limit Theorem & Law Of Large Numbers

4.F.1. A Central Limit Theorem

The full central limit theorem is discussed in section S4.C. Here, we deal with a simplified version concerning measurements. Consider the deviation from the average of N independent measurements of X,

$$Y_N = \frac{1}{N}\sum_{i=1}^{N} X_i - \langle x \rangle \quad = \frac{1}{N}\sum_{i=1}^{N} X_i - \frac{1}{N}\sum_{i=1}^{N}\langle x \rangle \quad = \sum_{i=1}^{N} Z_i \qquad (4.72)$$

where

$$Z_i = \frac{1}{N}\left(X_i - \langle x \rangle \right)$$

The characteristic function for Z is

$$f_Z\left(k;N\right) = \int_{-\infty}^{\infty} dz e^{ikz} P_Z\left(z\right) \quad = \int_{-\infty}^{\infty} dx P_X\left(x\right) \exp\left[i\frac{k}{N}\left(x - \langle x \rangle \right) \right]$$

$$= \left\langle \exp\left[i\frac{k}{N}\left(x - \langle x \rangle \right) \right] \right\rangle \qquad (4.73a)$$

Using

$$\exp\left[i\frac{k}{N}\left(x - \langle x \rangle \right) \right] = \sum_{n=0}^{\infty}\left(\frac{ik}{N} \right)^n \frac{\left(x - \langle x \rangle \right)^n}{n!}$$

$$= 1 + i\frac{k}{N}\left(x - \langle x \rangle \right) - \frac{1}{2}\left(\frac{k}{N} \right)^2 \left(x - \langle x \rangle \right)^2 + \cdots$$

eq(4.73a) becomes

$$f_Z\left(k;N\right) = 1 + i\frac{k}{N}\left\langle \left(x - \langle x \rangle \right) \right\rangle - \frac{1}{2}\left(\frac{k}{N} \right)^2 \left\langle \left(x - \langle x \rangle \right)^2 \right\rangle + \cdots$$

$$= 1 - \frac{1}{2}\left(\frac{k}{N} \right)^2 \sigma_X^2 + \cdots \qquad (4.73)$$

According to (4.40-1) and (4.72), we have

$$P_{Y_N}\left(y\right) = \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_N \delta\left[y - \frac{1}{N}\sum_{i=1}^{N}\left(x_i - \langle x \rangle \right) \right] P_X\left(x_1\right) \cdots P_X\left(x_N\right)$$

and

$$f_{Y_N}\left(k\right) = \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_N \exp\left[i\frac{k}{N}\sum_{j=1}^{N}\left(x_j - \langle x \rangle \right) \right] P_X\left(x_1\right) \cdots P_X\left(x_N\right)$$

$$= \prod_{j=1}^{N} \left\{ \int_{-\infty}^{\infty} dx_j \exp\left[i \frac{k}{N} \left(x_j - \langle x \rangle \right) \right] P_X \left(x_j \right) \right\}$$

$$= \prod_{j=1}^{N} \left\langle \exp\left[i \frac{k}{N} \left(x_j - \langle x \rangle \right) \right] \right\rangle \quad = \left\langle \exp\left[i \frac{k}{N} \left(x - \langle x \rangle \right) \right] \right\rangle^N$$

$$= \left[1 - \frac{1}{2} \left(\frac{k}{N} \right)^2 \sigma_X^2 + \cdots \right]^N \qquad \text{[(4.73) used]}$$

$$\to \exp\left(-\frac{k^2 \sigma_X^2}{2N} \right) \qquad \text{as} \qquad N \to \infty$$

Taking the inverse Fourier transform gives

$$P_{Y_N}(y) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-iky} f_{Y_N}(k)$$

$$\to \int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp\left(-iky - \frac{k^2 \sigma_X^2}{2N} \right) \qquad \text{as} \qquad N \to \infty$$

$$= \exp\left(-\frac{Ny^2}{2\sigma_X^2} \right) \int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp\left[-\frac{\sigma_X^2}{2N} \left(k + i \frac{Ny}{\sigma_X^2} \right)^2 \right]$$

$$= \sqrt{\frac{N}{2\pi\sigma_X^2}} \exp\left(-\frac{Ny^2}{2\sigma_X^2} \right) \qquad (4.75)$$

which is a Gaussian irregardless of the actual form of $P_X(x)$. This is known as the **central limit theorem**.

4.F.2. The Law Of Large Numbers

Tchebycheff Inequality

Consider the variance σ_X^2 of a random variable *X*:

$$\sigma_X^2 = \int_{-\infty}^{\infty} dx \left(x - \langle x \rangle \right)^2 P_X\left(x \right) \qquad (4.76)$$

Since the integrand is non-negative, we have

$$\sigma_X^2 \ge \left[\int_{-\infty}^{\langle x \rangle - \varepsilon} + \int_{\langle x \rangle + \varepsilon}^{\infty} \right] dx \left(x - \langle x \rangle \right)^2 P_X\left(x \right) \qquad (4.77)$$

where the interval $\left| x - \langle x \rangle \right| \le \varepsilon$ was taken out of the integration in (4.63).

For the intgrands in (4.64), $\left| x - \langle x \rangle \right| \ge \varepsilon$, so that

$$\sigma_X^2 \ge \varepsilon^2 \left[\int_{-\infty}^{\langle x \rangle - \varepsilon} + \int_{\langle x \rangle + \varepsilon}^{\infty} \right] dx P_X\left(x \right)$$

$$= \varepsilon^2 P\left[\left| x - \langle x \rangle \right| \ge \varepsilon \right] \qquad (4.78)$$

where $P\left[\left| x - \langle x \rangle \right| \ge \varepsilon \right]$ is the probability for *X* to deviate from $\langle x \rangle$ by more than ε.

Eq(4.65) can be rewritten as

$$P\left[\left| x - \langle x \rangle \right| \ge \varepsilon \right] \le \frac{\sigma_X^2}{\varepsilon^2} \qquad (4.79)$$

which is known as the **Tchebycheff inequality**.

Law of Large Numbers

Consider *N* independent measurements on *X*.

The mean value of the outcomes is defined as the random variable Y_N with values

$$y_N = \frac{1}{N} \sum_{j=1}^{N} x_j$$

Thus

$$\langle y_N \rangle = \frac{1}{N}\sum_{j=1}^{N}\langle x \rangle = \langle x \rangle$$

Since the measurements are independent, we have [see §4.D.3]

$$\sigma_{Y_N}^2 = \frac{1}{N^2}\sum_{j=1}^{N}\sigma_X^2 = \frac{\sigma_X^2}{N}$$

By the Tchebycheff inequality, we have

$$P\left[\left|y_N - \langle x \rangle\right| \geq \varepsilon\right] \leq \frac{\sigma_{Y_N}^2}{\varepsilon^2} = \frac{\sigma_X^2}{N\varepsilon^2} \qquad (4.80)$$

Hence

$$P\left[\left|y_N - \langle x \rangle\right| \geq \varepsilon\right] = 0 \qquad \text{as} \quad N \to \infty \qquad (4.81)$$

which is one expression of the **Law of Large Numbers**.