Special Section 4

S4.A. Lattice Random Walk

S4.A.1. One-Dimensional Lattice

S4.A.1.a. Periodic Lattice

Consider a random walker on a periodic 1-D lattice with $2N + 1$ sites labelled $-N, \ldots,$ $0, \ldots, N$. Note that for any integer m, site $l+m(2N+1)$ is the same as site l. Let $P_s(l)$ be the probability for finding the walker at site l at (discrete) time s.

Periodicity of the lattice means

$$P_s(l) = P_s\left[l + m(2N+1)\right]$$

Assuming that at each time step, the walker has an equal probability of taking a step to the left or to the right, the probability density for the ith step is

$$p(l_i) = \frac{1}{2}\left(\delta_{l_i,+1} + \delta_{l_i,-1}\right) \qquad (4.82)$$

where l_i is the displacement made at step i. If the walker starts at site $l = a$ at time $s = 0$, the total displacement after s steps is $l = a + \sum_{i=1}^{s} l_i$. Since the events are independent, we have

$$P_s(l) = \sum_{l_1=-N}^{N} \cdots \sum_{l_s=-N}^{N} \delta_{l,\ a+l_1+\cdots+l_s}\ p(l_1)\cdots p(l_s) \qquad (4.83)$$

Since $P_s(l)$ is spatially periodic, it can be written as a Fourier series,

$$P_s(l) = \frac{1}{2N+1}\sum_{n=-N}^{N} f_s(n)\exp\left(-\frac{2\pi i\, nl}{2N+1}\right) \qquad (4.84)$$

Using the identity

$$\frac{1}{2N+1}\sum_{l=-N}^{N}\exp\left(\pm\frac{2\pi i\,(n-m)\,l}{2N+1}\right) = \delta_{n,m} \qquad (4.84a)$$

the inverse of (4.84) is

$$f_s(n) = \sum_{l=-N}^{N} P_s(l)\exp\left(\frac{2\pi i\, nl}{2N+1}\right) \qquad (4.85)$$

Note that the Fourier amplitude $f_s(n)$ is also the characteristic function. Similarly, the Fourier transform of the transition probability $p(l)$ is

$$\lambda(n) = \sum_{i=-N}^{N} p(l)\exp\left(\frac{2\pi inl}{2N+1}\right)$$

$$= \frac{1}{2}\sum_{i=-N}^{N}\left(\delta_{l,+1} + \delta_{l,-1}\right)\exp\left(\frac{2\pi inl}{2N+1}\right) \qquad [(4.82)\text{ used}]$$

$$= \cos\left(\frac{2\pi n}{2N+1}\right) \tag{4.86}$$

Putting (4.83) into (4.85), we have

$$f_s(n) = \sum_{l=-N}^{N} \sum_{l_1=-N}^{N} \cdots \sum_{l_s=-N}^{N} \delta_{l,\ l_1+\cdots+l_s}\ p(l_1)\cdots p(l_s) \exp\left(\frac{2\pi i\ nl}{2N+1}\right)$$

$$= \sum_{l_1=-N}^{N} \cdots \sum_{l_s=-N}^{N} p(l_1)\cdots p(l_s)\ \exp\left(\frac{2\pi i\ n(a+l_1+\cdots+l_s)}{2N+1}\right)$$

$$= \exp\left(\frac{2\pi i\ na}{2N+1}\right) \sum_{l_1=-N}^{N} p(l_1)\exp\left(\frac{2\pi i\ nl_1}{2N+1}\right)\cdots \sum_{l_s=-N}^{N} p(l_s)\exp\left(\frac{2\pi i\ nl_s}{2N+1}\right)$$

$$= f_0(n)\left[\lambda(n)\right]^s \tag{4.87}$$

where

$$f_0(n) = \exp\left(\frac{2\pi i\ na}{2N+1}\right)$$

If the walker starts at site $l = a = 0$ at time $s = 0$, then $f_0(n) = 1$.

S4.A.1.b. Continuum

In the limit $N \to \infty$, we set $\phi = \frac{2\pi}{2N+1} n$ so that

$$\frac{1}{2N+1} \sum_{n=-N}^{N} \to \frac{1}{2N+1} \int_{-N}^{N} dn \simeq \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi$$

and (4.84) becomes

$$P_s(l) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi f_s(\phi) \exp(-il\phi) \tag{4.88}$$

while (4.85) gives

$$f_s(\phi) = \sum_{l=-\infty}^{\infty} P_s(l) \exp(il\phi) \tag{4.89}$$

For a single step *i*, (4.88) becomes

$$p(l_i) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \, \lambda(\phi) \exp(-il_i\phi) \tag{4.90}$$

where, according to the counterpart of (4.87), i.e.,

$$f_s(\phi) = f_0(\phi)\left[\lambda(\phi)\right]^s = f_0(\phi)\left[\cos\phi\right]^s \tag{4.91}$$

we have set $f_1(\phi) = \lambda(\phi)$. Putting (4.91) into (4.88) gives

$$P_s(l) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \; f_0(\phi)(\cos\phi)^s \exp(-il\phi)$$

$$= \frac{1}{2\pi}\left(\frac{1}{2}\right)^s \int_{-\pi}^{\pi} d\phi \exp(ia\phi)\left[\exp(i\phi) + \exp(-i\phi)\right]^s \exp(-il\phi)$$

$$= \frac{1}{2\pi}\left(\frac{1}{2}\right)^s \sum_{n=0}^{s} \frac{s!}{n!(s-n)!} \int_{-\pi}^{\pi} d\phi \exp\left[i(2n-s)\phi\right] \exp\left[-i(l-a)\phi\right]$$

$$= \left(\frac{1}{2}\right)^s \sum_{n=0}^{s} \frac{s!}{n!(s-n)!} \delta(2n-s-l+a)$$

$$= \begin{cases} \left(\frac{1}{2}\right)^s \dfrac{s!}{\left(\frac{s+l-a}{2}\right)!\left(\frac{s-l+a}{2}\right)!} & \text{for} \quad s-l+a = even \\ 0 & \quad otherwise \end{cases} \qquad (4.92)$$

S4.A.1.c. Generating Function

Consider the **generating function**

$$U(z,l) = \sum_{s=0}^{\infty} z^s P_s(l)$$

$$= \sum_{s=0}^{\infty} z^s \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi f_s(\phi) \exp(-il\phi) \qquad \text{[(4.88) used]}$$

$$= \sum_{s=0}^{\infty} z^s \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi f_0(\phi) \left[\lambda(\phi)\right]^s \exp(-il\phi) \qquad \text{[(4.91) used]}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \; f_0(\phi) \frac{\exp(-il\phi)}{1 - z\lambda(\phi)} \qquad (4.93)$$

where we've used the binomial expansion

$$(1 - z\lambda)^{-1} = \sum_{s=0}^{\infty} (z\lambda)^s$$

Thus,

$$P_s(l) = \lim_{z \to 0} \frac{1}{s!} \frac{\partial^s}{\partial z^s} U(z,l) \qquad (4.94)$$

S4.A.1.d. Exercise 4.12

Compute the generating function $U(z,0)$ for the probability to be at site $l=0$ at time *s*, given that the walker was at $l=0$ at time 0.

Answer

Since $l=0$ at time 0, we have $f_0(\phi)=1$, $\lambda(\phi)=\cos\phi$ and (4.93) gives

$$U(z,0)=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\phi\,\frac{1}{1-z\cos\phi}$$

$$=\frac{1}{2\pi}\cdot\frac{2}{\sqrt{1-z^2}}\tan^{-1}\left.\frac{\sqrt{1-z^2}\tan\frac{\phi}{2}}{1-z}\right|_{-\pi}^{\pi}$$

$$=\frac{1}{\sqrt{1-z^2}}$$

where we've used

$$\int\frac{dx}{a+b\cos x}=\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\frac{\sqrt{a^2-b^2}\tan\frac{x}{2}}{a+b}$$

S4.A.1.e. Escape Probability

Let $Q_s(l)$ be the probability that the walker arives at site *l* for the 1st time after *s* steps. The corresponding generating function is

$$V(z,l) = \sum_{s=1}^{\infty} z^s Q_s(l) \qquad (4.95)$$

In particular,

$$V(1,l) = \sum_{s=1}^{\infty} Q_s(l) \qquad (4.95a)$$

is the total probability that the walker reaches the site *l*.

Let the walker starts at $l = 0$ at $s = 0$ so that

$$P_0(l) = \delta_{l,0} \qquad (4.96)$$

Now, by the time he arrives at site *l* at time *s*, he may have visited it many times. Let $j \le s$ be the 1st time that he arrives at *l*. We can write

$$P_s(l) = \sum_{j=1}^{s} Q_j(l) P_{s-j}(0) \qquad (4.97)$$

where $P_{s-j}(0)$ is the probability that he visits the same site any number of times within a duration $s - j$. Putting these into (4.93), we have

$$\begin{aligned} U(z,l) &= \delta_{l,0} + \sum_{s=1}^{\infty} z^s P_s(l) \\ &= \delta_{l,0} + \sum_{s=1}^{\infty} \sum_{j=1}^{s} z^s Q_j(l) P_{s-j}(0) \qquad (4.98) \\ &= \delta_{l,0} + \sum_{j=1}^{\infty} \sum_{s=j}^{\infty} z^s Q_j(l) P_{s-j}(0) \\ &= \delta_{l,0} + \sum_{j=1}^{\infty} \sum_{t=0}^{\infty} z^{t+j} Q_j(l) P_t(0) \\ &= \delta_{l,0} + U(z,0) V(z,l) \qquad [(4.93,95) \text{ used}] \end{aligned}$$

$\Rightarrow$

$$V(z,l) = \frac{U(z,l) - \delta_{l,0}}{U(z,0)} \qquad (4.99)$$

By (4.95a), the total probability to return to the origin is

$$V(1,0) = \frac{U(1,0)-1}{U(1,0)} \quad = 1 - \frac{1}{U(1,0)} \qquad (4.100)$$

Hence, the total probability that the walker escapes from the origin is

$$P_{escape} = 1 - V(1,0) \quad = \frac{1}{U(1,0)} \qquad (4.101)$$

For the 1-D walker, $U(1,0) = \infty$ [see Exercise 4.12], so that $P_{escape} = 0$ and there is no escape.

S4.A.2. Random Walk in Higher Dimension

S4.B. Infinitely Divisible Distributions

S4.B.1. Gaussian Distribution

S4.B.2. Poisson Distribution

S4.B.3. Cauchy Distribution

S4.B.4. Levy Distribution

S4.C. The Central Limit Theorem

S4.C.1. Useful Inequalities

S4.C.2. Convergence to a Gaussian

S4.D. Weierstrass Random Walk

S4.D.1. Discrete One-Dimensional Random Walk

S4.D.2. Continuum Limit of One-Dimensional Discrete Random Walk

S4.D.3. Two-Dimensional Discrete Random Walk (Levy Flight)

S4.E. General Form of Infinitely Divisible Distributions

S4.E.1. Levy-Khintchine Formula

S4.E.2. Kolmogorov Formula