8. Order-Disorder Transitions and Renormalization Group

8.A. Introduction

8.B. Static Correlation Functions and Response Functions

8.B.1. General Relations

8.B.2. Application to the Ising Lattice

8.C. Scaling

8.C.1. Homogeneous Functions

Let

$$g(T,\mathbf{B}) = g_r(T,\mathbf{B}) + g_s(t,\mathbf{B})$$

where $t = \dfrac{T-T_C}{T_C}$ and the subscripts r and s stand for the regular and singular parts, respectively. Assuming g_s scales, we have

$$g_s(\lambda^p t, \lambda^q B) = \lambda g_s(t,B) \qquad [\ B = |\mathbf{B}|\ ]$$

Widom scaling: all other critical exponents can be expressed in terms of p and q.

1. Order Parameter

$$M(t, B=0) \propto (-t)^\beta$$

Note that $M = 0$ for $t > 0$ so that β is not defined there. Now,

$$M = -\frac{\partial g}{\partial B} \sim -\frac{\partial g_s}{\partial B}$$

$$\frac{\partial g_s(\lambda^p t, \lambda^q B)}{\partial B} = \lambda^q \frac{\partial g_s(\lambda^p t, \lambda^q B)}{\partial(\lambda^q B)} = -\lambda^q M(\lambda^p t, \lambda^q B)$$

$$= \lambda \frac{\partial g_s(t,B)}{\partial B} = -\lambda M(t,B)$$

Let

$$\lambda = (-t)^{-1/p} \qquad \text{and} \qquad B = 0$$

$\Rightarrow$

$$(-t)^{-q/p} M(-1,0) = (-t)^{-1/p} M(t,0)$$

$$\therefore \qquad M(t,0) \propto (-t)^{(1-q)/p} \qquad \Rightarrow \qquad \beta = \frac{1-q}{p}$$

2. Degree of Critical Isotherm

$$M(0,B) \propto |B|^{1/\delta} \operatorname{sgn} B$$

Again, from

$$\lambda^q M(\lambda^p t, \lambda^q B) = \lambda M(t,B)$$

and setting

$$\lambda = B^{-1/q} \qquad \text{and} \qquad t = 0$$

$\Rightarrow$

$$\frac{1}{B} M\left(0,1\right) = B^{-1/q} M\left(0,B\right)$$

$\therefore \qquad M\left(0,B\right) \propto B^{-1+q^{-1}} \qquad \Rightarrow \qquad \delta = \frac{q}{1-q}$

3. Susceptibility

$$\chi \propto \begin{cases} \left(-t\right)^{-\gamma'} \\ t^{-\gamma} \end{cases} \qquad \text{for} \qquad \begin{matrix} t < 0 \\ t > 0 \end{matrix}$$

$$\chi = \left(\frac{\partial M}{\partial B}\right)_t = -\left(\frac{\partial^2 g_s}{\partial B^2}\right)_t$$

Now,

$$\lambda^q \frac{\partial M\left(\lambda^p t, \lambda^q B\right)}{\partial B} = \lambda^{2q} \frac{\partial M\left(\lambda^p t, \lambda^q B\right)}{\partial\left(\lambda^q B\right)} = \lambda^{2q} \chi\left(\lambda^p t, \lambda^q B\right)$$

$$= \lambda \frac{\partial M\left(t,B\right)}{\partial B} = \lambda \chi\left(t,B\right)$$

Setting

$$\lambda = t^{-1/p} \qquad \text{and} \qquad B = 0$$

$\Rightarrow$

$$t^{-2q/p} \chi\left(1,0\right) = t^{-1/p} \chi\left(t,0\right)$$

$\therefore \qquad \chi\left(t,0\right) \propto t^{(1-2q)/p} \qquad \Rightarrow \qquad \gamma = \frac{2q-1}{p} = \gamma' \qquad \left(t \to -t \right)$

4. Heat Capacity

$$c_B \propto \begin{cases} \left(-t\right)^{-\alpha'} \\ t^{-\alpha} \end{cases} \qquad \text{for} \qquad \begin{matrix} t < 0 \\ t > 0 \end{matrix}$$

$$c_B = -T\left(\frac{\partial^2 g}{\partial T^2}\right)_B$$

Now,

$$T\frac{\partial^2 g_s\left(\lambda^p t,\lambda^q B\right)}{\partial t^2} = \lambda^{2p} T \frac{\partial^2 g_s\left(\lambda^p t,\lambda^q B\right)}{\partial\left(\lambda^p t\right)^2} \quad = -\lambda^{2p} c_B\left(\lambda^p t,\lambda^q B\right)$$

$$= T\lambda \frac{\partial^2 g_s\left(t,B\right)}{\partial t^2} \quad = -\lambda c_B\left(t,B\right)$$

Setting

$\lambda = t^{-1/p}$ and $B = 0$

$\Rightarrow$

$$t^{-2} c_B\left(1,0\right) = t^{-1/p} c_B\left(t,0\right)$$

$\therefore \quad c_B\left(t,0\right) \propto t^{p^{-1}-2} \quad \Rightarrow \quad \alpha = 2 - \frac{1}{p} = \alpha'$

Summary

$$\alpha = 2 - \frac{1}{p} = \alpha'$$

$$\beta = \frac{1-q}{p}$$

$$\gamma = \frac{2q-1}{p} = \gamma'$$

$$\delta = \frac{q}{1-q}$$

or

$$\frac{1}{p} = 2 - \alpha \qquad (1)$$

$$\frac{1}{p} - \frac{q}{p} = \beta \qquad (2)$$

$$\frac{1}{p} - 2\frac{q}{p} = -\gamma \qquad (3)$$

$$\frac{1}{1-q} = \delta + 1 \qquad (4)$$

so that

(2),(3) $\Rightarrow \quad \frac{1}{p} = 2\beta + \gamma$

With (1) $\Rightarrow \quad 2 - \alpha = 2\beta + \gamma \quad \Rightarrow \quad \alpha + 2\beta + \gamma = 2$

(2)*(4) $\Rightarrow \quad \frac{1}{p} = \beta\left(\delta + 1\right)$

With (1) $\Rightarrow \quad \beta\left(\delta + 1\right) = 2 - \alpha \quad \Rightarrow \quad \alpha + \beta\left(\delta + 1\right) = 2$

(4): $$\delta - 1 = \frac{1}{1-q} - 2 = \frac{-1+2q}{1-q} \qquad (5)$$

(2)*(5) ⇒ $$\frac{-1+2q}{p} = \beta\left(\delta - 1\right) = \gamma \qquad \text{(from (3))}$$

Since all critical exponents are expressed in terms of *p* and *q*, only 2 of them are independent. Note that

$$p = \frac{1}{\beta\left(\delta+1\right)} \qquad q = \frac{\delta}{\delta+1}$$

8.C.2. Widom Scaling

Let

$$g(T,\mathbf{B}) = g_r(T,\mathbf{B}) + g_s(t,\mathbf{B})$$

where $t = \frac{T - T_C}{T_C}$ and the subscripts r and s stand for the regular and singular parts, respectively. Assuming g_s scales, we have

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Setting

$$\lambda = t^{-1/p} \qquad \text{and} \qquad B = 0$$

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$$t^{-2q/p} \chi\left(1,0\right) = t^{-1/p} \chi\left(t,0\right)$$

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$$= T\lambda \frac{\partial^2 g_s\left(t, B\right)}{\partial t^2} \quad = -\lambda c_B\left(t, B\right)$$

Setting

$$\lambda = t^{-1/p} \qquad \text{and} \qquad B = 0$$

$\Rightarrow$

$$t^{-2} c_B\left(1,0\right) = t^{-1/p} c_B\left(t,0\right)$$

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Summary

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Since all critical exponents are expressed in terms of *p* and *q*, only 2 of them are independent. Note that

$$p = \frac{1}{\beta\left(\delta+1\right)} \qquad q = \frac{\delta}{\delta+1}$$

8.C.3. Kadanoff Scaling

Consider the d-dim nearest neighbor (n.n.) Ising model

$$H[s] = -K \sum_{\langle i,j \rangle} s_i s_j - B \sum_i s_i \qquad (1)$$

where $\langle i, j \rangle$ denotes n.n. pairs. Thus, for a lattice of N sites, each of which having γ n.n., the sum $\sum_{\langle i,j \rangle}$ contains $\frac{1}{2}\gamma N$ terms.

We now divide the lattice into blocks with linear dimensions $La \ll \xi$, where a is the lattice constant and ξ the correlation length. Thus, the total number of such blocks is $\frac{N}{L^d}$ while the number of sites (spins) in each block is L^d. Let the total spin in block I be $S'_I = \sum_{i \in I} s_i$. Since $s_i = \pm 1$, we have $-L^d \le S'_I \le L^d$. Thus, we can write $S'_I = Z S_I$ where $S_I = \pm 1$ and $0 \le Z \le L^d$. Furthermore, the condition $La \ll \xi$ means that the spins in each block are more or less aligned with each other so that $Z \simeq L^d$. Eq(1) can thus be written in terms of the blocks as

$$H[S_L] = -K_L \sum_{\langle I,J \rangle} S_I S_J - B_L \sum_I S_I \qquad (2)$$

where $I = 1, \cdots, \frac{N}{L^d}$ and the sum over $\langle I, J \rangle$ gives $\frac{1}{2}\gamma \frac{N}{L^d}$ terms.

Now, $H[s]$ and $H[S_L]$ has the same functional form. Since there are L^d sites in each block, we have

$$g(t_L, B_L) = L^d g(t, B)$$

$$\xi_L(t_L, B_L) = \frac{\xi(t,B)}{L}$$

Now, let $t_L = L^x t$. Since reducing the length scale should move the system away from the critical point, we have $t_L > t$ so that $x > 0$. From

$$B \sum_{i=1}^{N} s_i = B \sum_{I=1}^{N/L^d} S'_I \quad = BZ \sum_I S_I \quad = B_L \sum_I S_I$$

we have

$$B_L = BZ \equiv L^y B$$

Since $Z \le L^d$, we have $L^y \le L^d$ or $y \le d$. Hence

$$g\left(L^x t, L^y B\right) = L^d g(t,B)$$

which, in comparison with the Widom scaling

$$g_s\left(\lambda^p t, \lambda^q B\right) = \lambda g_s(t,B)$$

gives

$$L^d = \lambda \qquad \text{so that} \qquad \frac{x}{d} = p \qquad \frac{y}{d} = q$$

$\Rightarrow \qquad q < 1$

Since $q = \dfrac{\delta}{\delta+1}$, we have $\delta > 0$, as expected. Now,

$$C(r_L, t_L) = \langle S_I S_J \rangle - \langle S_I \rangle \langle S_J \rangle \quad = \frac{1}{Z^2}\left[\langle S'_I S'_J \rangle - \langle S'_I \rangle \langle S'_J \rangle\right]$$

$$= \frac{1}{Z^2}\sum_{i\in I}\sum_{j\in J}\left[\langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle\right] \quad = \frac{1}{Z^2}\sum_{i\in I}\sum_{j\in J} C(r,t)$$

$$= \frac{1}{Z^2}\left(L^d\right)^2 C(r,t) \quad = L^{2(d-y)} C(r,t)$$

Comparing with

$$C(r_L, t_L) = C\left(L^{-1} r, t\, L^x\right) \qquad \left(r_L = \frac{r}{L} \right)$$

we have

$$C(r,t) = L^{2(y-d)} C\left(L^{-1} r, t\, L^x\right)$$

$$= \left(\frac{r}{a}\right)^{2(y-d)} C\left(a, t\left(\frac{r}{a}\right)^x\right) \qquad (r = La)$$

Define $\xi = t^{-\nu}$. (Note that $\nu = \dfrac{1}{2}$ for mean field theories).

$$C(\mathbf{r}, t) = C\left(\frac{r}{\xi}\right) \quad = C\left(r t^{\nu}\right) \quad = C\left(t\, r^x\right) \quad = C\left(t^{1/x} r\right)$$

$\Rightarrow \qquad \nu = \dfrac{1}{x} \quad = \dfrac{1}{pd}$

For $t = 0$ (at critical point),

$$C(r,0)=\left(\frac{r}{a}\right)^{2(y-d)}C(a,0) \quad \propto \left(\frac{r}{a}\right)^{2(y-d)} \quad \propto r^{2-d-\eta}$$

$$\Rightarrow \qquad 2(y-d)=2-d-\eta$$

$$=2(dg-d) \qquad\qquad (\quad y=dq \quad)$$

$$\therefore \qquad \eta=2-d-2d(q-1) \quad =2-d(2q-1) \quad =2-d\left(\frac{2\delta}{\delta+1}-1\right)$$

$$=2-d\,\frac{(\delta-1)}{\delta+1}$$

Using the Widom scaling $\beta(\delta-1)=\gamma$, we have $\delta+1=2+\dfrac{\gamma}{\beta}$ so that

$$\eta=2-d\,\frac{\gamma}{\beta}\cdot\frac{\beta}{2\beta+\gamma} \quad =2-\frac{d\gamma}{2\beta+\gamma}$$

Also,

$$\nu=\frac{1}{pd} \quad =\frac{2-\alpha}{d} \qquad\qquad (\quad p=\frac{1}{2-\alpha} \quad)$$

8.D. Microscopic Calculations of Critical Exponents

8.D.1. Renormalization Group

Ref: T.Niemeijer, J.M.J.van Leeuwen, in "Phase Transitions & Critical Phenomena", Vol.6, ed. C.Domb, M.S.Green (76)

The most general form of the (effective) spin hamiltonian is

$$K(\kappa, s, N) = \kappa_0 + \kappa_1 \sum_i s_i + \kappa_2^{(1)} \sum_{i,j}^{(1)} s_i s_j + \kappa_2^{(2)} \sum_{i,j}^{(2)} s_i s_j + \kappa_3^{(1)} \sum_{i,j,k}^{(1)} s_i s_j s_k + \cdots$$

where $\Sigma^{(n)}$ is a sum over the nearest n neighbors. Also, $s = \{s_i\}$ and $\kappa = \{\kappa_m\}$, where κ_m is the coupling constant between blocks of m spins. Note that $K = \beta H$.

For the Ising model,

$$\kappa_1 = -\beta B$$

$$\kappa_2 = -\beta J$$

$$\kappa_m = 0 \qquad \text{otherwise}$$

Now,

$$Z(\kappa, N) = \sum_s \exp\left[-K(\kappa, s, N)\right]$$

Define the block spin $S_I = \sum_{i \in I} \sigma_i$, where σ_i is the spin inside block *I*. (Note that σ can itself be a block spin). Hence,

$$Z(\kappa, N) = \sum_{S_L, \sigma_L} \exp\left[-K(\kappa, S_L, \sigma_L, N)\right]$$

$$= \sum_{S_L} \exp\left[-K\left(\kappa_L, S_L, \frac{N}{L^d}\right)\right]$$

$$= Z\left(\kappa_L, \frac{N}{L^d}\right)$$

where S_L is the shorthand for $\{S_I \text{ , } I = block\ of\ \ size\ L^d\}$ and $\sigma_L = \{\sigma_i \text{ , } i \in I\}$.

$$g(\kappa) = \lim_{N \to \infty} \frac{1}{N} \ln Z(\kappa, N) \quad = \lim_{N \to \infty} \frac{1}{N} \ln Z\left(\kappa_L, \frac{N}{L^d}\right) \quad = L^{-d} g(\kappa_L)$$

Let $\kappa_L = T(\kappa)$. Note that κ and κ_L are vectors while the transformation *T* is non-linear but form preserving. Thus,

$$\kappa_{2L} = T(\kappa_L) \qquad \ldots \qquad \kappa_{nL} = T\left(\kappa_{(n-1)L}\right)$$

At the critical point, $\kappa = \kappa^*$ so that

$$\kappa^* = T\left(\kappa^*\right) \qquad \text{[R.G. eq.]}$$

i.e., κ is a fixed point of T at the critical point.

Let

$$\delta\kappa_L = \kappa_L - \kappa^* \qquad \delta\kappa = \kappa - \kappa^*$$

From $\kappa_L = T\left(\kappa\right)$, we have

$$\kappa_L = \kappa^* + \nabla_\kappa \kappa_L\big|_{\kappa=\kappa^*} \cdot \left(\kappa - \kappa^*\right) + \cdots$$

or

$$\delta\kappa_L \simeq \nabla_\kappa \kappa_L\big|_{\kappa=\kappa^*} \cdot \delta\kappa \quad = A \cdot \delta\kappa$$

where A is a matrix with elements $a_{ij} = \left.\dfrac{\partial\kappa_{Li}}{\partial\kappa_j}\right|_{\kappa=\kappa^*}$. Note that in general, $a_{ij} \neq a_{ji}$ so that the left and right eigenvectors can be different and the eigenvalues may not be all real. Let

$$SAS^{-1} = \Lambda = diagonal$$

so that

$$S\delta\kappa_L = SAS^{-1}S\delta\kappa$$

$$\Rightarrow \qquad \delta u_L = \Lambda\delta u$$

where $\delta u_L = S\delta\kappa_L$ and $\delta u = S\delta\kappa$. In matrix form, we have

$$\begin{pmatrix} \delta u_{L1} \\ \vdots \\ \delta u_{LM} \end{pmatrix} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_M \end{pmatrix} \begin{pmatrix} \delta u_1 \\ \vdots \\ \delta u_M \end{pmatrix}$$

where M is the dimension of the coupling constant space. Hence,

$$\delta u_{NL} = \Lambda^N \delta u$$

or $\qquad \delta u_{NLi} = \lambda_i^N \delta u_i$

Note that a curve going through κ^* and satisfying $\delta u_i = \sum_j S_{ij} \delta\kappa_j$ is called an **eigencurve** of the eigenvalue λ_i. Points on an eigencurve thus

1. move away from κ^* if $\lambda_i > 1 \Rightarrow$ relevant (δu_i physical)

2. move toward κ^* if $\lambda_i < 1 \Rightarrow$ critical (δu_i irrelevant)

Hence, near a critical point,

$$g(\delta\kappa) = g(\delta u) = L^{-d} g(\delta\kappa_L) = L^{-d} g(\Lambda \delta u)$$

or

$$g_s(\delta u_1, \cdots, \delta u_M) = L^{-d} g_s(\lambda_1 \delta u_1, \cdots, \lambda_M \delta u_M)$$

To compare with the Widom scaling,

$$g(t, B) = \frac{1}{\lambda} g(\lambda^p t, \lambda^q B)$$

we set

$$\delta u_1 = t \qquad \delta u_2 = B$$

$\Rightarrow$

$$\lambda = L^d$$

$$\lambda_1 = \lambda^p = L^{dp} \qquad \Rightarrow \qquad p = \frac{1}{d} \cdot \frac{\ln \lambda_1}{\ln L}$$

$$\lambda_2 = \lambda^q = L^{dq} \qquad \Rightarrow \qquad q = \frac{1}{d} \cdot \frac{\ln \lambda_2}{\ln L}$$

All other critical exponents can be expressed in terms of λ_i.

8.D.2. Exercise 8.1: Triangular Lattice

$$Z(\mathbf{\kappa}) = \sum_s \exp\left[-H(\mathbf{\kappa}, s)\right]$$

$$H(\mathbf{\kappa}, s) = -\kappa \sum_{i \neq j}^{(1)} s_i s_j - B \sum_i s_i \qquad (\ s_i = \pm 1 \ )$$

$$\mathbf{\kappa} = (-\kappa, -B, 0, \cdots)$$

Consider the block spin

$$s_I = sign\left(s_1^I + s_2^I + s_3^I\right)$$

where s_i^I is the *i*th spin in block *I*.

α	s_1^I	s_2^I	s_3^I	s_I	σ_I^α
1	1	1	1	1	3
2	1	1	–1	1	1
3	1	–1	1	1	1
4'	1	–1	–1	–1	1
4	–1	1	1	1	1
3'	–1	1	–1	–1	1
2'	–1	–1	1	–1	1
1'	–1	–1	–1	–1	3

A given spin configuration

$$s = \{s_i\} \qquad \text{where } i = 1, \cdots, N \text{ and } s_i = \pm 1$$

can be specified in block spin terms

$$s_L = \{s_I, \alpha_I\} \qquad I = 1, \cdots, \frac{N}{3} \quad \text{and} \quad s_I = \pm 1, \quad \alpha_I = 1,2,3,4$$

$$= \{s_I, \sigma_I\} \qquad \sigma_I = \begin{cases} 3 \\ 1 \end{cases}$$

$$Z(\mathbf{\kappa}) = \sum_s \exp\left[-H(\mathbf{\kappa}, s)\right] = \sum_{s_L} \exp\left[-H(\mathbf{\kappa}, s_L)\right]$$

$$= \sum_{\{s_I, \alpha_I\}} \exp\left[-H\left(\mathbf{\kappa}, \{s_I, \alpha_I\}\right)\right]$$

$$= \sum_{s_1=-1}^{1} \sum_{\alpha_1=1}^{4} \cdots \sum_{s_{N/3}=-1}^{1} \sum_{\alpha_{N/3}=1}^{4} \exp\left[-H\left(\mathbf{\kappa}, s_1, \alpha_1, \cdots, s_{N/3}, \alpha_{N/3}\right)\right]$$

$$\equiv \sum_{\{s_I\}} \exp\left[-H\left(\mathbf{\kappa}_L, \{s_I\}\right)\right]$$

Thus, for a given configuration $\{s_I\} = \{s_1, \cdots, s_{N/3}\}$,

$$\exp\left[-H\left(\mathbf{\kappa}_L, \{s_I\}\right)\right] = \sum_{\{\alpha_I\}} \exp\left[-H\left(\mathbf{\kappa}, \{s_I, \alpha_I\}\right)\right]$$

$$= \sum_{\alpha_1=1}^{4} \cdots \sum_{\alpha_{N/3}=1}^{4} \exp\left[-H\left(\mathbf{\kappa}, \{s_I, \alpha_I\}\right)\right]$$

Now,

$$H\left(\mathbf{\kappa}, s\right) = -\kappa \sum_{i \neq j}^{(1)} s_i s_j - B \sum_i s_i$$

$$= -\kappa \sum_I \sum_{i \neq j \in I}^{(1)} s_i s_j - \kappa \sum_{I \neq J} \sum_{i \in I,\, j \in J}^{(1)} s_i s_j - B \sum_I \sum_{i \in I} s_i$$

$$= H_0 + V$$

where

$$H_0 = -\kappa \sum_I \sum_{i \neq j \in I}^{(1)} s_i s_j$$

$$V = -\kappa \sum_{I \neq J} \sum_{i \in I,\, j \in J}^{(1)} s_i s_j - B \sum_I \sum_{i \in I} s_i$$

Note that H_0 doesn't contain inter-block interactions. Consider the Ith term in H_0,

$$h_{0I} = -\kappa \sum_{i \neq j \in I}^{(1)} s_i s_j \quad = -\kappa\ \mu_I^{\alpha} \left(s_I\right)^2 \quad = -\kappa\ \mu_I^{\alpha}$$

α	s_1^I	s_2^I	s_3^I	$\mu = s_1 s_2 + s_2 s_3 + s_3 s_1$
1	1	1	1	$1+1+1=3$
2	1	1	–1	$1-1-1=-1$
3	1	–1	1	$-1-1+1=-1$
4	–1	1	1	$-1+1-1=-1$

Hence, for a given configuration $\{s_i\} = \{s_I, \alpha_I\}$,

$$H_0 = H_0\left(\mathbf{\kappa}, \{s_i\}\right) \quad = H_0\left(\mathbf{\kappa}, \{s_I, \alpha_I\}\right) \quad = -\kappa \sum_I \mu_I^{\alpha_I} \left(s_I\right)^2$$

where $\mu^{\alpha} = \begin{cases} 3 \\ -1 \end{cases}$ for $s_I = \pm 1$.

$\Rightarrow$

$$\exp\left[-H\left(\boldsymbol{\kappa}_L,\{s_I\}\right)\right]=\sum_{\{\alpha_I\}}\exp\left[-H\left(\boldsymbol{\kappa},\{s_I,\alpha_I\}\right)\right]$$

$$=\sum_{\{\alpha_I\}}\exp\left[-H_0\left(\boldsymbol{\kappa},\{s_I,\alpha_I\}\right)\right]\exp\left[-V\left(\boldsymbol{\kappa},\{s_I,\alpha_I\}\right)\right]$$

Now, let

$$\langle A\rangle_0\equiv\frac{1}{Z_0}\sum_{\{\alpha_I\}}\exp\left(-H_0\right)A$$

or

$$\left\langle A\left(\{s_I\}\right)\right\rangle_0\equiv\frac{1}{Z_0\left(\{s_I\}\right)}\sum_{\{\alpha_I\}}\exp\left[-H_0\left(\{s_I,\alpha_I\}\right)\right]A\left(\{s_I,\alpha_I\}\right)$$

where

$$Z_0\left(\{s_I\}\right)=\sum_{\{\alpha_I\}}\exp\left[-H_0\left(\{s_I,\alpha_I\}\right)\right]$$

$\Rightarrow$

$$\exp\left[-H\left(\boldsymbol{\kappa}_L,\{s_I\}\right)\right]=Z_0\left(\boldsymbol{\kappa},\{s_I\}\right)\left\langle\exp\left(-V\right)\right\rangle_0$$

where $\langle\cdots\rangle_0$ is an average over $\{\alpha_I\}$ only, i.e., no average over $\{s_I\}$. Now,

$$Z_0\left(\boldsymbol{\kappa},\{s_I\}\right)=\sum_{\{\alpha_I\}}\exp\left[-H_0\left(\boldsymbol{\kappa},\{s_I,\alpha_I\}\right)\right]$$

$$=\sum_{\{\alpha_I\}}\exp\left[\kappa\sum_I\mu_I^{\alpha_I}\left(s_I\right)^2\right]$$

$$=\prod_{I=1}^{M}\sum_{\alpha=1}^{4}\exp\left(\kappa\,\mu^{\alpha}\right)\qquad\qquad(M=\frac{N}{3})$$

$$=\prod_I\left(e^{3\kappa}+3e^{-\kappa}\right)$$

$$=\left[z_0\left(\boldsymbol{\kappa}\right)\right]^M$$

where

$$z_0\left(\boldsymbol{\kappa}\right)=e^{3\kappa}+3e^{-\kappa}$$

Thus,

$$\exp\left[-H\left(\boldsymbol{\kappa}_L,\{s_I\}\right)\right]=\left[z_0\left(\boldsymbol{\kappa}\right)\right]^M\left\langle e^{-V}\right\rangle_0$$

In the cumulant expansion scheme,

$$\left\langle e^{-V}\right\rangle_0 = \exp\left\{-\left\langle V\right\rangle_0 + \frac{1}{2}\left[\left\langle V^2\right\rangle_0 - \left\langle V\right\rangle_0^2\right] + \cdots\right\}$$

so that

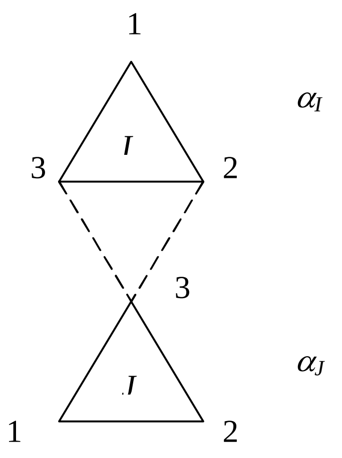
$$-H\left(\boldsymbol{\kappa}_L, \{s_I\}\right) = M \ln z_0\left(\boldsymbol{\kappa}\right) - \left\langle V\right\rangle_0 + \frac{1}{2}\left[\left\langle V^2\right\rangle_0 - \left\langle V\right\rangle_0^2\right] + \cdots$$

Now, consider the (I, J) term in the 1st sum of

$$V = -\kappa \sum_{I \neq J} \sum_{i \in I,\, j \in J}{}^{(1)} s_i s_j - B \sum_I \sum_{i \in I} s_i$$

i.e.,

$$v_{IJ} = -\kappa \sum_{i \in I,\, j \in J}{}^{(1)} s_i s_j$$



For the case shown in the figure, we have

$$v_{IJ} = -\kappa\left(s_1^I s_3^J + s_2^I s_3^J\right) = -\kappa\left(s_1^I + s_2^I\right) s_3^J$$

Hence,

$$\left\langle v_{IJ}\right\rangle_0 = -\kappa\left(\left\langle s_1^I s_3^J\right\rangle_0 + \left\langle s_2^I s_3^J\right\rangle_0\right)$$

Now,

$$\left\langle a^I b^J\right\rangle_0 = \frac{1}{Z_0} \sum_{\{\alpha_L\}} a^I b^J \exp\left(\kappa \sum_L \mu_L^{\alpha_L}\right)$$

$$= \frac{1}{z_0^2} \sum_{\alpha_I} \sum_{\alpha_J} a^I b^J \exp\left[\kappa\left(\mu_I^{\alpha_I} + \mu_J^{\alpha_J}\right)\right]$$

$$= \left\langle a^I\right\rangle_0 \left\langle b^J\right\rangle_0$$

Hence,

$$\left\langle v_{IJ}\right\rangle_0 = -\kappa\left(\left\langle s_1^I\right\rangle_0 + \left\langle s_2^I\right\rangle_0\right)\left\langle s_3^J\right\rangle_0 \quad = -2\kappa\left\langle s_i^I\right\rangle_0\left\langle s_i^J\right\rangle_0 \qquad (\ i \text{ arbitrary })$$

Note that the last equality is valid for all block orientations.

$$\left\langle s_i^I\right\rangle_0 = \frac{1}{z_0}\sum_{\alpha=1}^{4} s_I s_i \exp\left(\kappa\mu^\alpha\right)$$
$$= \frac{s_I}{z_0}\left(e^{3\kappa} + 2e^{-\kappa} - e^{-\kappa}\right) \quad = \frac{s_I}{z_0}\left(e^{3\kappa} + e^{-\kappa}\right)$$
$$= s_I\,\frac{e^{3\kappa} + e^{-\kappa}}{e^{3\kappa} + 3e^{-\kappa}}$$

$$\therefore \qquad \left\langle v_{IJ}\right\rangle_0 = -2\kappa\left(\frac{e^{3\kappa} + e^{-\kappa}}{e^{3\kappa} + 3e^{-\kappa}}\right)^2 s_I s_J$$

Hence,

$$\left\langle V\right\rangle = -2\kappa\left(\frac{e^{3\kappa} + e^{-\kappa}}{e^{3\kappa} + 3e^{-\kappa}}\right)^2 \sum_{I\neq J} s_I s_J - 3B\,\frac{e^{3\kappa} + e^{-\kappa}}{e^{3\kappa} + 3e^{-\kappa}}\sum_I s_I$$

where the factor of 3 comes from $\sum_{i\in I} = \sum_{i=1}^{3}$. Thus, to order $\left\langle V\right\rangle_0$,

$$H\left(\boldsymbol{\kappa}_L, \{s_L\}, B_L\right) \simeq -M\ln z_0\left(\boldsymbol{\kappa}\right) - 2\kappa\left(\frac{e^{3\kappa} + e^{-\kappa}}{e^{3\kappa} + 3e^{-\kappa}}\right)^2 \sum_{I\neq J} s_I s_J - 3B\,\frac{e^{3\kappa} + e^{-\kappa}}{e^{3\kappa} + 3e^{-\kappa}}\sum_I s_I$$

The singular part is

$$H_S\left(\boldsymbol{\kappa}_L, \{s_L\}, B_L\right) = -\kappa_L\sum_{I\neq J} s_I s_J - B_L\sum_I s_I$$

with

$$\kappa_L = 2\kappa\left(\frac{e^{3\kappa} + e^{-\kappa}}{e^{3\kappa} + 3e^{-\kappa}}\right)^2 \qquad (1)$$

$$B_L = 3B\,\frac{e^{3\kappa} + e^{-\kappa}}{e^{3\kappa} + 3e^{-\kappa}} \qquad (2)$$

(1) $\Rightarrow$ $\qquad K^* = 0$

or $\qquad 1 = 2\left(\frac{e^{3\kappa} + e^{-\kappa}}{e^{3\kappa} + 3e^{-\kappa}}\right)^2 \qquad (3)$

(2) $\Rightarrow$ $\qquad B^* = 0$

or $\qquad 1 = 3\,\frac{e^{3\kappa} + e^{-\kappa}}{e^{3\kappa} + 3e^{-\kappa}} \qquad (4)$

Since we are interested only in the critical point at $B = 0$, the fixed point of interest is $K^* = 0$ or the solution of (3). The linearized stability matrix equation is

$$\begin{pmatrix} \delta\kappa_L \\ \delta B_L \end{pmatrix} = \begin{pmatrix} \frac{\partial\kappa_L}{\partial\kappa} & \frac{\partial\kappa_L}{\partial B} \\ \frac{\partial B_L}{\partial\kappa} & \frac{\partial B_L}{\partial B} \end{pmatrix}_{\kappa=\kappa^*, B=B^*} \begin{pmatrix} \delta\kappa \\ \delta B \end{pmatrix} = A \begin{pmatrix} \delta\kappa \\ \delta B \end{pmatrix}$$

Now,

$$\frac{\partial\kappa_L}{\partial\kappa} = 2\left(\frac{e^{3\kappa} + e^{-\kappa}}{e^{3\kappa} + 3e^{-\kappa}}\right)^2$$

$$+4\kappa\left(\frac{e^{3\kappa} + e^{-\kappa}}{e^{3\kappa} + 3e^{-\kappa}}\right)\frac{\left(e^{3\kappa} + 3e^{-\kappa}\right)\left(3e^{3\kappa} - e^{-\kappa}\right) - \left(3e^{3\kappa} - 3e^{-\kappa}\right)\left(e^{3\kappa} + e^{-\kappa}\right)}{\left(e^{3\kappa} + 3e^{-\kappa}\right)^2}$$

$$\frac{\partial\kappa_L}{\partial B} = 0$$

$$\frac{\partial B_L}{\partial\kappa} = 3B\frac{\left(e^{3\kappa} + 3e^{-\kappa}\right)\left(3e^{3\kappa} - e^{-\kappa}\right) - \left(3e^{3\kappa} - 3e^{-\kappa}\right)\left(e^{3\kappa} + e^{-\kappa}\right)}{\left(e^{3\kappa} + 3e^{-\kappa}\right)^2}$$

$$\frac{\partial B_L}{\partial B} = 3\left(\frac{e^{3\kappa} + e^{-\kappa}}{e^{3\kappa} + 3e^{-\kappa}}\right)$$

For $K^* = 0$ and $B^* = 0$, we have

$$\frac{\partial\kappa_L}{\partial\kappa} = 2\left(\frac{2}{4}\right)^2 = \frac{1}{2}$$

$$\frac{\partial\kappa_L}{\partial B} = 0$$

$$\frac{\partial B_L}{\partial\kappa} = 0$$

$$\frac{\partial B_L}{\partial B} = 3\left(\frac{2}{4}\right) = \frac{3}{2}$$

so that

$$A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix}$$

$\Rightarrow \qquad \lambda_\kappa = \frac{1}{2} \qquad$ and $\qquad \lambda_B = \frac{3}{2}$

Note that $\kappa = \frac{J}{k_B T}$ where J is the exchange integral. Hence,

$$K^* = 0 \quad \Rightarrow \quad T_C \to \infty \quad \text{for finite } J.$$

Since $\lambda_\kappa = \frac{1}{2} < 1$ it is irrelevant. For $B = 0$, λ_B is discarded. Thus, the fixed point has no relevant eigenvalue and is therefore unphysical.

Solution to (3) is

$$\frac{e^{3\kappa} + e^{-\kappa}}{e^{3\kappa} + 3e^{-\kappa}} = \pm\frac{1}{\sqrt{2}}$$

$$\Rightarrow \qquad \left(1 \mp \frac{1}{\sqrt{2}}\right) e^{3\kappa} + \left(1 \mp \frac{3}{\sqrt{2}}\right) e^{-\kappa} = 0$$

$$e^{4\kappa} = -\frac{1 \mp \frac{3}{\sqrt{2}}}{1 \mp \frac{1}{\sqrt{2}}} = \frac{\pm 3 - \sqrt{2}}{\sqrt{2} \mp 1} = \left(\pm 3 - \sqrt{2}\right)\left(\sqrt{2} \pm 1\right) = 3 - 2 \pm 2\sqrt{2}$$

$$= 1 \pm 2\sqrt{2}$$

Since $e^{4\kappa} > 0$ for real κ, we have

$$e^{4\kappa} = 1 + 2\sqrt{2}$$

so that

$$\kappa^* = \frac{1}{4}\ln\left(1 + 2\sqrt{2}\right) \simeq 0.34$$

and

$$\frac{e^{3\kappa^*} + e^{-\kappa^*}}{e^{3\kappa^*} + 3e^{-\kappa^*}} = \frac{1}{\sqrt{2}}$$

Hence

$$\frac{\partial \kappa_L}{\partial \kappa} = 2\left(\frac{1}{\sqrt{2}}\right)^2 + 4 \times 0.34 \times \frac{1}{\sqrt{2}} \times \left(\cdots\right) \simeq 1.62$$

$$\frac{\partial \kappa_L}{\partial B} = 0$$

$$\frac{\partial B_L}{\partial \kappa} = 0$$

$$\frac{\partial B_L}{\partial B} = 3\left(\frac{1}{\sqrt{2}}\right) \simeq 2.12$$

so that

$$A = \begin{pmatrix} 1.62 & 0 \\ 0 & 2.12 \end{pmatrix}$$

$\Rightarrow$ $\lambda_\kappa = 1.62$ and $\lambda_B = 2.12$

Thus, both eigenvalues are relevant. (They also remain unchanged for $B \neq 0$.)

For 2-dim,

$$p = \frac{\ln \lambda_\kappa}{d \ln \lambda_B} = \frac{\ln 1.62}{2 \ln 1.73} = 0.44$$

$$q = \frac{\ln \lambda_B}{d \ln \lambda_\kappa} = 0.68$$

$$\alpha = 2 - \frac{1}{p} = -0.27$$

$$\delta = \frac{q}{1-q} = 2.1$$

These are to be compared with the exact solutions

$\alpha_{exact} = 0$ and $\delta_{exact} = 15$

which implies

$(\lambda_\kappa)_{exact} = 1.73$ and $(\lambda_B)_{exact} = 2.80$

Thus, λ_κ is quite good but λ_B is badly off.