Special Section 8

S8.A. Critical Exponents for the S^4 Model

Consider the generalized Ising Model:

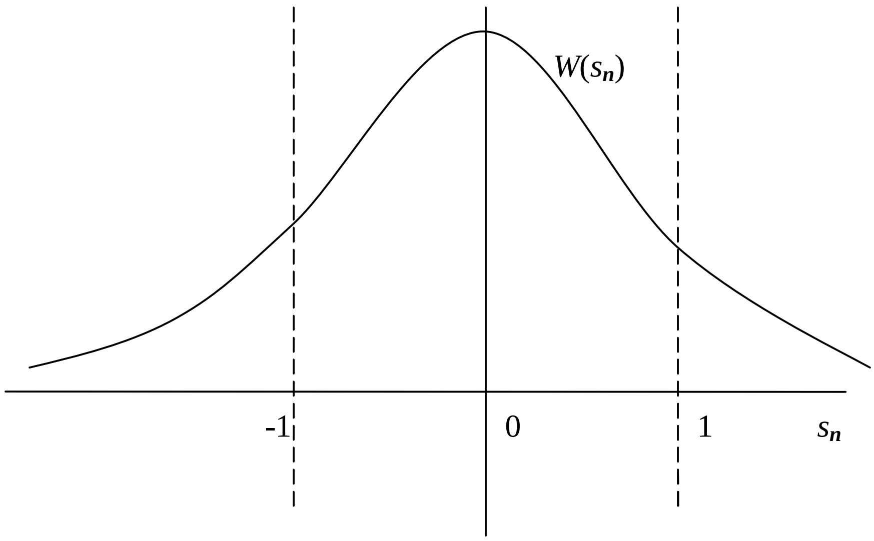
$$Z = \prod_m \int_{-\infty}^{\infty} ds_m \, W[s] \exp\left(\kappa \sum_{\mathbf{n},}{}' s_{\mathbf{n}} s_{\mathbf{n}+} \right)$$

To get the Gaussian model, we set

$$W[s] = \exp\left(-\frac{b}{2} \sum_{\mathbf{n}} s_{\mathbf{n}}^2 \right) = \prod_{\mathbf{n}} e\,\mathrm{xp}\left(-\frac{b}{2} s_{\mathbf{n}}^2 \right) = \prod_{\mathbf{n}} W(s_{\mathbf{n}})$$

where

$$W(s_{\mathbf{n}}) = \exp\left(-\frac{b}{2} s_{\mathbf{n}}^2 \right)$$



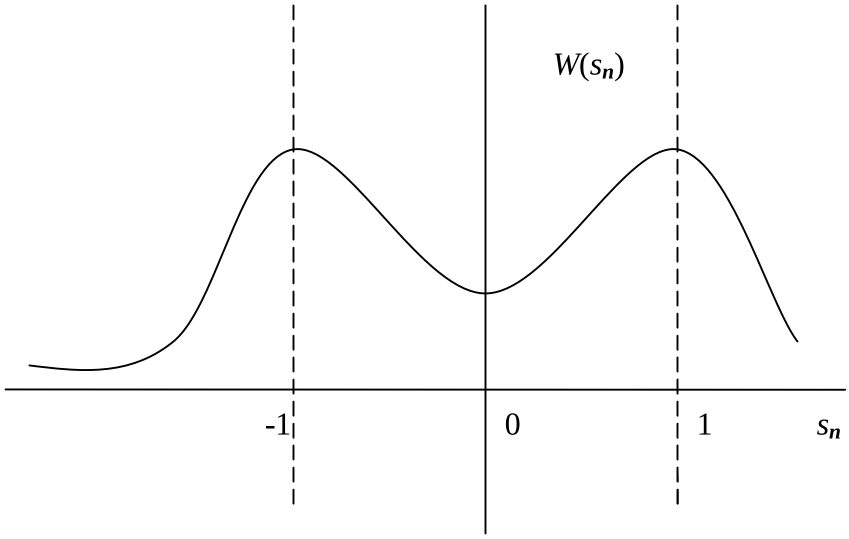
To get the S^4 model, we set

$$W(s_{\mathbf{n}}) = \exp\left(-\frac{b}{2} s_{\mathbf{n}}^2 - u s_{\mathbf{n}}^4 \right) \qquad \text{with} \qquad u \ge 0$$

For $b = -4u$, we have

$$W(s_{\mathbf{n}}) = \exp\left[-u\left(s_{\mathbf{n}}^4 - 2 s_{\mathbf{n}}^2 \right) \right] = \exp\left[-u\left(s_{\mathbf{n}}^2 - 1 \right)^2 + u \right]$$

which should be a better approximation to the Ising model.



Let $B = 0$ and $\boldsymbol{\kappa} = (\kappa, b, u)$, then

$$Z = Z\left(\kappa, b, u, \{s_{\mathbf{n}}\}\right)$$

$$= \prod_{m} \int_{-\infty}^{\infty} ds_m \ \exp\left[\kappa {\sum_{\mathbf{n},}}' s_{\mathbf{n}} s_{\mathbf{n}+} \ - \sum_{\mathbf{n}} \left(\frac{b}{2} s^2 + u s^4\right)\right]$$

$$= \prod_{m} \int_{-\infty}^{\infty} ds_m \ \exp\left[-H\left(\kappa, b, u, \{s_{\mathbf{n}}\}\right)\right]$$

where

$$H\left(\kappa, b, u, \{s_{\mathbf{n}}\}\right) = -\kappa {\sum_{\mathbf{n},}}' s_{\mathbf{n}} s_{\mathbf{n}+} \ + \sum_{\mathbf{n}} \left(\frac{b}{2} s^2 + u s^4\right)$$

Now, from the Gaussian model,

$$-\kappa {\sum_{\mathbf{n},}}' s_{\mathbf{n}} s_{\mathbf{n}+} \ + \sum_{\mathbf{n}} \frac{b}{2} s^2 = \frac{1}{2(2\pi)^d} \int_{B.Z.} d^d k \left|s'(\mathbf{k})\right|^2 \left(r + k^2\right)$$

where $s'(\mathbf{k}) = s(\mathbf{k})\sqrt{\kappa a^{2-d}}$. Similarly,

$$\sum_{\mathbf{n}} s_{\mathbf{n}}^4 = \sum_{\mathbf{n}} \frac{1}{(2\pi)^{4d}} \int_{B.Z.} d^d k_1 \int_{B.Z.} d^d k_2 \int_{B.Z.} d^d k_3 \int_{B.Z.} d^d k_4$$

$$\times s(\mathbf{k}_1) s(\mathbf{k}_2) s(\mathbf{k}_3) s(\mathbf{k}_4) \exp\left[i(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) \cdot \mathbf{n} a\right]$$

Using

$$\sum_{\mathbf{n}} \exp(i\mathbf{k}\cdot\mathbf{n}a) = \left(\frac{2\pi}{a}\right)^d \delta(\mathbf{k})$$

$\Rightarrow$

$$\sum_{\mathbf{n}} s_{\mathbf{n}}^4 = \frac{1}{(2\pi)^{4d}}\left(\frac{2\pi}{a}\right)^d \int_{B.Z.} d^d k_1 \int_{B.Z.} d^d k_2 \int_{B.Z.} d^d k_3 \int_{B.Z.} d^d k_4$$

$$\times\, s(\mathbf{k}_1)s(\mathbf{k}_2)s(\mathbf{k}_3)s(\mathbf{k}_4)\,\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)$$

$$= \frac{1}{(2\pi)^{3d}}\cdot\frac{1}{\kappa^2 a^{4-d}} \int d^d k_1 \int d^d k_2 \int d^d k_3 \int d^d k_4$$

$$\times\, s'(\mathbf{k}_1)s'(\mathbf{k}_2)s'(\mathbf{k}_3)s'(\mathbf{k}_4)\,\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)$$

where

$$\frac{1}{a^d}\cdot\frac{1}{\kappa^2 a^{2(2-d)}} = \frac{1}{\kappa^2 a^{4-d}}$$

Let

$$u' = \frac{u}{\kappa^2 a^{4-d}}$$

we have

$$H(r,u',\{s'\}) = \frac{1}{2(2\pi)^d}\int d^d k\, s'(\mathbf{k})s'(-\mathbf{k})(r+k^2)$$

$$+\frac{u'}{(2\pi)^{3d}} \int d^d k_1 \int d^d k_2 \int d^d k_3 \int d^d k_4$$

$$\times\, s'(\mathbf{k}_1)s'(\mathbf{k}_2)s'(\mathbf{k}_3)s'(\mathbf{k}_4)\,\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)$$

Let

$$H = H_{long} + H_{short} + V$$

where

$$H_{long} = H_{long}(r,\{S_L\}) \quad = \frac{1}{2(2\pi)^d}\int_{long} d^d k\, |S_L(\mathbf{k})|^2 (r+k^2)$$

$$H_{short} = H_{short}(r,\{\sigma_L\}) \quad = \frac{1}{2(2\pi)^d}\int_{short} d^d k\, |\sigma_L(\mathbf{k})|^2 (r+k^2)$$

$$V = V\left(u',\{S_L,\sigma_L\}\right) = \frac{u'}{(2\pi)^{3d}} \int d^d k_1 \int d^d k_2 \int d^d k_3 \int d^d k_4$$

$$\times s'(\mathbf{k}_1) s'(\mathbf{k}_2) s'(\mathbf{k}_3) s'(\mathbf{k}_4)\, \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4)$$

Note that S_L corresponds to block spin S_I and σ_L to internal degrees of freedom σ_I. We'll treat V as a perturbation and use the cumulant expansion approximation. Thus, we start with introducing the average over the internal degrees of freedom

$$\langle A \rangle_0 = \frac{1}{Z_0} \int D\sigma_L \exp\left(-H_{short}\right) A$$

or

$$\langle A\{S_L\} \rangle_0 = \frac{1}{Z_0} \int D\sigma_L \exp\left[-H_{short}\left(r,\{\sigma_L\}\right)\right] A\left(\{S_L,\sigma_L\}\right)$$

where

$$Z_0 = \int D\sigma_L \exp\left[-H_{short}\left(r,\{\sigma_L\}\right)\right] = Z_0(r)$$

Note that $\sum_{\{\alpha_L\}} \to \int D\sigma_L$ when comparing with the triangular lattice treatment.

$$Z(r,u') = \int DS_L \int D\sigma_L\, e^{-H}$$

$$= \int DS_L \int D\sigma_L \exp\left(-H_{long} - H_{short} - V\right)$$

$$= \int DS_L \int D\sigma_L \exp\left[-H_{long}\left(r,\{S_L\}\right) - H_{short}\left(r,\{\sigma_L\}\right) - V\left(u',\{\sigma_L,S_L\}\right)\right]$$

Using

$$\left\langle V\left(u',\{S_L\}\right)\right\rangle_0 = \frac{1}{Z_0} \int D\sigma_L \exp\left[-H_{short}\left(r,\{\sigma_L\}\right)\right] V\left(u',\{S_L,\sigma_L\}\right)$$

and

$$\left\langle \exp\left[-V\left(u',\{S_L\}\right)\right]\right\rangle_0 = \frac{1}{Z_0} \int D\sigma_L \exp\left[-H_{short}\left(r,\{\sigma_L\}\right) - V\left(u',\{S_L,\sigma_L\}\right)\right]$$

we have

$$Z(r,u') = Z_0(r) \int DS_L \exp\left[-H_{long}\left(r,\{S_L\}\right)\right] \left\langle \exp\left[-V\left(u',\{S_L\}\right)\right]\right\rangle_0$$

In the cumulant expansion

$$\left\langle e^{-V}\right\rangle_0 = \exp\left[-\langle V\rangle_0 + \frac{1}{2}\left(\langle V^2\rangle_0 - \langle V\rangle_0^2\right) + \cdots\right]$$

so that

$$Z(r,u') \simeq Z_0(r)\int DS_L \ \exp\left[-H_{long}(r,\{S_L\}) - \langle V\rangle_0 + \frac{1}{2}\left(\langle V^2\rangle_0 - \langle V\rangle_0^2\right) + \cdots\right]$$

Note that $Z_0(r)$ contributes only to the regular part of g and hence can be dropped.

Thus, the singular part of Z is

$$Z_S(r,u') \simeq \int DS_L \ \exp\left[-H(r,u',\{S_L\})\right]$$

where

$$H(r,u',\{S_L\}) = H_{long}(r,\{S_L\}) + \left\langle V(u',\{S_L\})\right\rangle_0$$

$$-\frac{1}{2}\left(\left\langle V(u',\{S_L\})^2\right\rangle_0 - \left\langle V(u',\{S_L\})\right\rangle_0^2\right) + \cdots$$

$$= \frac{1}{2(2\pi)^d}\int_{long} d^dk \left|S_L(\mathbf{k})\right|^2 (r+k^2) + \langle V\rangle_0 - \frac{1}{2}\left(\langle V^2\rangle_0 - \langle V\rangle_0^2\right) + \cdots$$

Note that H corresponds to the Kadanoff transformation result.

To bring H back to the original hamiltonian, we need a scale transformation. Now,

$$\langle V\rangle_0 = \frac{1}{Z_0}\int D\sigma_L \ \exp(-H_{short})V$$

where

$$V = \frac{u'}{(2\pi)^{3d}}\int d^dk_1 \int d^dk_2 \int d^dk_3 \int d^dk_4$$

$$\times s'(\mathbf{k}_1)s'(\mathbf{k}_2)s'(\mathbf{k}_3)s'(\mathbf{k}_4)\,\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)$$

If we write

$$\int_{B.Z.} d^dk_i \ s'(\mathbf{k}_i) = \int_{long} d^dk_i \ S_L(\mathbf{k}_i) + \int_{short} d^dk_i \ \sigma_L(\mathbf{k}_i) \qquad i = 1,\cdots,4$$

then V becomes a sum of $2^4 = 16$ terms. However, since $\exp(-H_{short})$ is invariant under $\sigma_L \to -\sigma_L$, we have

$$\frac{1}{Z_0}\int D\sigma_L \ \exp(-H_{short}) \times (odd \ \ power \ of \ \ \sigma_L) = 0$$

Also, $\mathbf{k}_i$ are simply dummy variables so that terms with the same powers of σ_L and S_L are equal after the $\mathbf{k}_i$ integrations. Since

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b_2 + 4ab^3 + b^4$$

we have

$$V=\frac{u'}{(2\pi)^{3d}}\left\{\int_{long} d^d k_1 \int_{long} d^d k_2 \int_{long} d^d k_3 \int_{long} d^d k_4\right.$$

$$\times S_L(\mathbf{k}_1) S_L(\mathbf{k}_2) S_L(\mathbf{k}_3) S_L(\mathbf{k}_4)\,\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)$$

$$+6\int_{long} d^d k_1 \int_{long} d^d k_2 \int_{short} d^d k_3 \int_{short} d^d k_4$$

$$\times S_L(\mathbf{k}_1) S_L(\mathbf{k}_2)\sigma_L(\mathbf{k}_3)\sigma_L(\mathbf{k}_4)\,\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)$$

$$+\int_{short} d^d k_1 \int_{short} d^d k_2 \int_{short} d^d k_3 \int_{short} d^d k_4$$

$$\left.\times \sigma_L(\mathbf{k}_1)\sigma_L(\mathbf{k}_2)\sigma_L(\mathbf{k}_3)\sigma_L(\mathbf{k}_4)\,\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\right\}$$

The last term can be dropped since it contributes only to the regular part. Hence,

$$\langle V_S\rangle_0=\frac{u'}{(2\pi)^{3d}}\left\{\int_{long} d^d k_1 \int_{long} d^d k_2 \int_{long} d^d k_3 \int_{long} d^d k_4\right.$$

$$\times S_L(\mathbf{k}_1) S_L(\mathbf{k}_2) S_L(\mathbf{k}_3) S_L(\mathbf{k}_4)\,\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)$$

$$+6\int_{long} d^d k_1 \int_{long} d^d k_2 \int_{short} d^d k_3 \int_{short} d^d k_4$$

$$\left.\times S_L(\mathbf{k}_1) S_L(\mathbf{k}_2)\langle\sigma_L(\mathbf{k}_3)\sigma_L(\mathbf{k}_4)\rangle_0\,\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\right\}$$

where

$$\langle\sigma_L(\mathbf{k}_3)\sigma_L(\mathbf{k}_4)\rangle_0=\frac{1}{Z_0}\int D\sigma_L\,\exp(-H_{short})\,\sigma_L(\mathbf{k}_3)\sigma_L(\mathbf{k}_4)$$

Since

$$H_{short}=\frac{1}{2(2\pi)^d}\int_{short} d^d k\,|\sigma_L(\mathbf{k})|^2(r+k^2)$$

therefore, $\langle\cdots\rangle_0$ is a Gaussian average (see Einstein fluctuation). To get

$\langle\sigma_L(\mathbf{k}_3)\sigma_L(\mathbf{k}_4)\rangle_0$, we 1st write H_{short} in discrete sum using

$$\frac{\tilde{V}}{(2\pi)^d}\int d^d k\rightarrow\sum_{\mathbf{k}}$$ where $\tilde{V}$ is the volume of the system.

Hence,

$$H_{short} = \frac{1}{2\tilde{V}} \sum_{\mathbf{k} \in short} \left|\sigma_L(\mathbf{k})\right|^2 \left(r + k^2\right)$$

and

$$\left\langle \sigma_L(\mathbf{k}_3)\sigma_L(\mathbf{k}_4)\right\rangle_0 = \frac{\tilde{V}}{r + k_3^2}\delta_{\mathbf{k}_3+\mathbf{k}_4,\mathbf{0}} \;\; \rightarrow \;\; \frac{(2\pi)^d}{r + k_3^2}\delta(\mathbf{k}_3 + \mathbf{k}_4)$$

Therefore,

$$\left\langle V_S \right\rangle_0 = \frac{u'}{(2\pi)^{3d}} \left\{ \int_{long} d^d k_1 \int_{long} d^d k_2 \int_{long} d^d k_3 \int_{long} d^d k_4 \right.$$

$$\times S_L(\mathbf{k}_1) S_L(\mathbf{k}_2) S_L(\mathbf{k}_3) S_L(\mathbf{k}_4)\, \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4)$$

$$\left. + 6(2\pi)^d \int_{long} d^d k_1 \int_{long} d^d k_2 \int_{short} d^d k_3\, S_L(\mathbf{k}_1) S_L(\mathbf{k}_2) \frac{1}{r + k_3^2}\, \delta(\mathbf{k}_1 + \mathbf{k}_2) \right\}$$

$$= \frac{u'}{(2\pi)^{3d}} \int_{long} d^d k_1 \int_{long} d^d k_2 \int_{long} d^d k_3 \int_{long} d^d k_4$$

$$\times S_L(\mathbf{k}_1) S_L(\mathbf{k}_2) S_L(\mathbf{k}_3) S_L(\mathbf{k}_4)\, \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4)$$

$$+ \frac{6u'}{(2\pi)^{2d}} \int_{long} d^d k_1 \left|S_L(\mathbf{k}_1)\right|^2 \int_{short} d^d k_3 \frac{1}{r + k_3^2}$$

Thus, $\left\langle V_S \right\rangle_0$ has the same form as *H*. However, it doesn't contribute to the quartic term. Now,

$$\left\langle V^2 \right\rangle_0 = \frac{1}{Z_0} \int D\sigma_L \exp\left(-H_{short}\right) V \cdot V$$

$$= \frac{1}{Z_0} \frac{u'}{(2\pi)^{6d}} \int D\sigma_L \exp\left(-H_{short}\right)$$

$$\times \int d^d k_1 \int d^d k_2 \int d^d k_3 \int d^d k_4\; s'(\mathbf{k}_1) s'(\mathbf{k}_2) s'(\mathbf{k}_3) s'(\mathbf{k}_4)\, \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4)$$

$$\times \int d^d k_5 \int d^d k_6 \int d^d k_7 \int d^d k_8\; s'(\mathbf{k}_5) s'(\mathbf{k}_6) s'(\mathbf{k}_7) s'(\mathbf{k}_8)\, \delta(\mathbf{k}_5 + \mathbf{k}_6 + \mathbf{k}_7 + \mathbf{k}_8)$$

Again, we write

$$\int_{B.Z.} d^d k_i\; s'(\mathbf{k}_i) = \int_{long} d^d k_i\; S_L(\mathbf{k}_i) + \int_{short} d^d k_i\; \sigma_L(\mathbf{k}_i) \qquad i = 1, \cdots, 8$$

The structure of $\left\langle V^2 \right\rangle_0$ is

$$\left(\sigma + S\right)^4 \left(\sigma + S\right)^4$$

$$= \left(\sigma^4 + 4\sigma^3 S + 6\sigma^2 S^2 + 4\sigma S^3 + S^4\right)\left(\sigma^4 + 4\sigma^3 S + 6\sigma^2 S^2 + 4\sigma S^3 + S^4\right)$$

Note that a term like $6\sigma^2 S^2$ really means 6 terms, i.e.,

$$\sigma_1\sigma_2 S_3 S_4 + \sigma_1 S_2 S_3 \sigma_4 + \sigma_1 S_2 \sigma_3 S_4 + S_1 \sigma_2 \sigma_3 S_4 + S_1 S_2 \sigma_3 \sigma_4 + S_1 \sigma_2 S_3 \sigma_4$$

where $S_i = S\left(\mathbf{k}_i\right)$ etc.

The quartic contribution of $\left\langle V^2 \right\rangle_0$ must consist of terms of products of 4*S* and 4σ.

Hence, we need only consider terms

$$\sigma^4 S^4 + S^4 \sigma^4 + 16\left(\sigma^3 S \sigma S^3 + \sigma S^3 \sigma^3 S\right) + 36\sigma^2 S^2 \sigma^2 S^2$$

where the arguments of each term are $\left(\mathbf{k}_1, \cdots, \mathbf{k}_8\right)$. Thus, for example,

$$\sigma^3 S \sigma S^3 = \left(\sigma^3 S\right)_{1\cdots 4}\left(\sigma S^3\right)_{5\cdots 8} = \left(\sigma^3 S\right)_I \left(\sigma S^3\right)_{II}$$

where *I* and *II* refers to quantities from the 1st and 2nd *V*, respectively. Note that spins belong to the same *V* are governed by the same delta function.

Now, terms from $\sigma^4 S^4 + S^4 \sigma^4$ have σ coming from the same *V*. After the averaging, the will be the same as the corresponding terms from $\left\langle V \right\rangle_0^2$. Hence, they don't contribute to the 2nd cumulant.

Also, terms like $\sigma^3 S \sigma S^3$ have odd powers of spin from each *V*. By symmetry, they vanish upon averaging. Hence, only the 36 terms of the form $\left(\sigma^2 S^2\right)_I \left(\sigma^2 S^2\right)_{II}$ contribute. Thus,

$$\frac{1}{2}\left(\left\langle V^2 \right\rangle_0 - \left\langle V \right\rangle_0^2\right)_{quartic} = \frac{1}{2} \cdot \frac{36}{Z_0} \int D\sigma_L \exp\left(-H_{short}\right)$$

$$\times \frac{u'^2}{\left(2\pi\right)^{6d}} \int_{long} d^d k_1 \int_{long} d^d k_2 \int_{short} d^d k_3 \int_{short} d^d k_4 \int_{short} d^d k_5 \int_{short} d^d k_6 \int_{long} d^d k_7 \int_{long} d^d k_8$$

$$\times \delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\,\delta(\mathbf{k}_5+\mathbf{k}_6+\mathbf{k}_7+\mathbf{k}_8)$$

$$\times S_L(\mathbf{k}_1)S_L(\mathbf{k}_2)S_L(\mathbf{k}_7)S_L(\mathbf{k}_8)\,\sigma_L(\mathbf{k}_3)\sigma_L(\mathbf{k}_4)\sigma_L(\mathbf{k}_5)\sigma_L(\mathbf{k}_6)$$

Now,

$$\langle\sigma_3\sigma_4\sigma_5\sigma_6\rangle_0=\langle\sigma_3\sigma_4\rangle_0\langle\sigma_5\sigma_6\rangle_0+\langle\sigma_3\sigma_5\rangle_0\langle\sigma_4\sigma_6\rangle_0+\langle\sigma_3\sigma_6\rangle_0\langle\sigma_4\sigma_5\rangle_0$$

The 1st term on the right can be dropped since it's cancelled by $\langle V\rangle_0^2$.

$$\langle\sigma_3\sigma_5\rangle_0\langle\sigma_4\sigma_6\rangle_0=\frac{(2\pi)^{2d}}{(r+k_3^2)(r+k_4^2)}\delta(\mathbf{k}_3+\mathbf{k}_5)\,\delta(\mathbf{k}_4+\mathbf{k}_6)$$

$$\langle\sigma_3\sigma_6\rangle_0\langle\sigma_4\sigma_5\rangle_0=\frac{(2\pi)^{2d}}{(r+k_3^2)(r+k_4^2)}\delta(\mathbf{k}_3+\mathbf{k}_6)\,\delta(\mathbf{k}_4+\mathbf{k}_5)$$

Thus, these 2 terms give the same contributions so that

$$\frac{1}{2}\left(\langle V^2\rangle_0-\langle V\rangle_0^2\right)_{quartic}$$

$$=36\frac{u'^2}{(2\pi)^{4d}}\int_{long}d^dk_1\int_{long}d^dk_2\int_{short}d^dk_3\int_{short}d^dk_4\int_{long}d^dk_7\int_{long}d^dk_8$$

$$\times \delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\,\delta(\mathbf{k}_7+\mathbf{k}_8-\mathbf{k}_3-\mathbf{k}_4)$$

$$\times\frac{1}{(r+k_3^2)(r+k_4^2)}S_L(\mathbf{k}_1)S_L(\mathbf{k}_2)S_L(\mathbf{k}_7)S_L(\mathbf{k}_8)$$

Integrating over $\mathbf{k}_4$ means

$$r+k_4^2\to r+(\mathbf{k}_7+\mathbf{k}_8-\mathbf{k}_3)^2$$

$$\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\;\to\;\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_7+\mathbf{k}_8)$$

Relabeling

$\mathbf{k}_7\to\mathbf{k}_3$, $\quad\mathbf{k}_8\to\mathbf{k}_4$, $\quad\mathbf{k}_3\to\mathbf{k}$

we have

$$\frac{1}{2}\left(\langle V^2\rangle_0-\langle V\rangle_0^2\right)_{quartic}$$

$$=36\frac{u'^2}{(2\pi)^{4d}}\int_{long}d^dk_1\int_{long}d^dk_2\int_{long}d^dk_3\int_{long}d^dk_4\;\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)$$

$$\times S_L(\mathbf{k}_1)S_L(\mathbf{k}_2)S_L(\mathbf{k}_3)S_L(\mathbf{k}_4)\int_{short} d^dk\,\frac{1}{(r+k^2)\left[r+(\mathbf{k}_3+\mathbf{k}_4-\mathbf{k})^2\right]}$$

Putting everything together, we have

$$H(r,u',\{S_L\})=\frac{1}{2(2\pi)^d}\int_{long} d^dk\,|S_L(\mathbf{k})|^2(r+k^2)$$

$$+\frac{u'}{(2\pi)^{3d}}\int_{long} d^dk_1\int_{long} d^dk_2\int_{long} d^dk_3\int_{long} d^dk_4$$

$$\times S_L(\mathbf{k}_1)S_L(\mathbf{k}_2)S_L(\mathbf{k}_3)S_L(\mathbf{k}_4)\,\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)$$

$$+\frac{6u'}{(2\pi)^{2d}}\int_{long} d^dk_1|S_L(\mathbf{k}_1)|^2\int_{short} d^dk_3\,\frac{1}{r+k_3^2}$$

$$-36\frac{u'^2}{(2\pi)^{4d}}\int_{long} d^dk_1\int_{long} d^dk_2\int_{long} d^dk_3\int_{long} d^dk_4\,\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)$$

$$\times S_L(\mathbf{k}_1)S_L(\mathbf{k}_2)S_L(\mathbf{k}_3)S_L(\mathbf{k}_4)\int_{short} d^dk\,\frac{1}{(r+k^2)\left[r+(\mathbf{k}_3+\mathbf{k}_4-\mathbf{k})^2\right]}$$

$$\simeq\frac{1}{2(2\pi)^d}\int_{long} d^dk\,|S_L(\mathbf{k})|^2\left[r+k^2+\frac{12u'}{(2\pi)^d}\int_{short} d^dk_1\,\frac{1}{r+k_1^2}\right]$$

$$+\frac{1}{(2\pi)^{3d}}\int_{long} d^dk_1\int_{long} d^dk_2\int_{long} d^dk_3\int_{long} d^dk_4$$

$$\times S_L(\mathbf{k}_1)S_L(\mathbf{k}_2)S_L(\mathbf{k}_3)S_L(\mathbf{k}_4)\,\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)$$

$$\times\left\{u'-36\frac{u'^2}{(2\pi)^d}\int_{short} d^dk\,\frac{1}{(r+k^2)\left[r+(\mathbf{k}_3+\mathbf{k}_4-\mathbf{k})^2\right]}\right\}$$

Note that we've neglected all quadratic and other terms from $\frac{1}{2}\left(\langle V^2\rangle_0-\langle V\rangle_0^2\right)$.

To bring H_S back to the same form as *H*, we need a scale transformation. Let

$k_L = Lk$

$$S_L\left(\frac{\mathbf{k}_L}{L}\right)=zS_L\left(\mathbf{k}_L\right)$$

we have

$$H_S\left(r,u',\{S_L\}\right)$$

$$\simeq\frac{1}{2(2\pi)^d}\int_{B.Z.}d^dk_L\,\frac{z^2}{L^d}\left|S_L\left(\mathbf{k}\right)\right|^2\left[r+\left(\frac{k_L}{L}\right)^2+\frac{12u'}{(2\pi)^d}\int_{short}d^dk_1\,\frac{1}{r+k_1^2}\right]$$

$$+\frac{1}{(2\pi)^{3d}}\int_{B.Z.}d^dk_{L1}\int_{B.Z.}d^dk_{L2}\int_{B.Z.}d^dk_{L3}\int_{B.Z.}d^dk_{L4}$$

$$\times\frac{z^4}{L^{4d}}S_L\left(\mathbf{k}_{L1}\right)S_L\left(\mathbf{k}_{L2}\right)S_L\left(\mathbf{k}_{L3}\right)S_L\left(\mathbf{k}_{L4}\right)L^d\ \delta\left(\mathbf{k}_{L1}+\mathbf{k}_{L2}+\mathbf{k}_{L3}+\mathbf{k}_{L4}\right)$$

$$\times\left\{u'-36\frac{u'^2}{(2\pi)^d}\int_{short}d^dk\,\frac{1}{\left(r+k^2\right)\left[r+\left(\frac{\mathbf{k}_{L3}+\mathbf{k}_{L4}}{L}-\mathbf{k}\right)^2\right]}\right\}$$

$$\equiv H_S\left(r_L,u_L,\{S_L\}\right)$$

$\Rightarrow$

$$r_L=\frac{z^2}{L^d}\left[r+\frac{12u'}{(2\pi)^d}\int_{short}d^dk_1\,\frac{1}{r+k_1^2}\right]$$

$$u_L=\frac{z^4}{L^{3d}}\left\{u'-36\frac{u'^2}{(2\pi)^d}\int_{short}d^dk\,\frac{1}{\left(r+k^2\right)^2}\right\}$$

where we've set $\mathbf{k}_{L3}+\mathbf{k}_{L4}=0$. Using $\frac{z^2}{L^dL^2}=1$, we have

$$r_L=L^2\left[r+\frac{12u'}{(2\pi)^d}\int_{short}d^dk\,\frac{1}{r+k^2}\right]$$

$$u_L=\frac{1}{L^{d-4}}\left\{u'-36\frac{u'^2}{(2\pi)^d}\int_{short}d^dk\,\frac{1}{\left(r+k^2\right)^2}\right\}$$

where

$$\frac{1}{(2\pi)^d}\int_{short} d^d k = \begin{cases} \dfrac{1}{(2\pi)^3}\cdot 4\pi \displaystyle\int_{\pi/La}^{\pi/a} dk\, k^2 \\ \dfrac{1}{(2\pi)^2}\cdot 2\pi \displaystyle\int_{\pi/La}^{\pi/a} dk\, k \\ \dfrac{1}{2\pi}\cdot 2 \displaystyle\int_{\pi/La}^{\pi/a} dk \end{cases} \qquad \text{for} \quad d = \begin{cases} 3 \\ 2 \\ 1 \end{cases}$$

Using

$$\frac{d}{dx}\int_{y(x)}^{c} dy\; f(y) = -\frac{dy}{dx}\, f(y) \qquad\qquad [\; c \text{ is a constant} \;]$$

we have

$$\frac{d}{dL}\int_{short} d^d k \,\frac{1}{\left(r+k^2\right)^n} = \begin{pmatrix} 4\pi \\ 2\pi \\ 2 \end{pmatrix} \frac{\pi}{aL^2}\cdot \frac{\left(\dfrac{\pi}{La}\right)^{d-1}}{\left[r+\left(\dfrac{\pi}{La}\right)^2\right]^n}$$

$$= \begin{pmatrix} 4\pi \\ 2\pi \\ 2 \end{pmatrix}\left(\frac{\pi}{a}\right)^d \frac{1}{L^{2+d}}\cdot \frac{1}{\left[r+\left(\dfrac{\pi}{La}\right)^2\right]^n}$$

Now, let

$$A(L,d) = \frac{12}{(2\pi)^d}\begin{pmatrix} 4\pi \\ 2\pi \\ 2 \end{pmatrix}\left(\frac{\pi}{a}\right)^d \frac{1}{L^{1+d}}$$

we have

$$\frac{dr_L}{dL} = 2L\left[r + \frac{12u'}{(2\pi)^d}\int_{short} d^d k\,\frac{1}{r+k^2}\right] + L^2\,\frac{12u'}{(2\pi)^d}\frac{\partial}{\partial L}\int_{short} d^d k\,\frac{1}{r+k^2}$$

$$= \frac{2r_L}{L} + \frac{A(L,d)}{L\left[r_L + \left(\dfrac{\pi}{La}\right)^2\right]}\, u'$$

Similarly,

$$\frac{du_L}{dL} = (4-d)\frac{u_L}{L} - \frac{3A(L,d)}{L^{d-3}\left[r_L + \left(\dfrac{\pi}{La}\right)^2\right]^2}\, u'^2$$

Let $t = \ln L$

$$\Rightarrow \qquad L = e^t \qquad dt = \frac{dL}{L} \qquad L\frac{d}{dL} = \frac{d}{dt}$$

so that

$$\frac{dr_L}{dt} = 2r_L + \frac{A\left(e^t, d\right)}{\left[r_L + \left(\frac{\pi}{e^t a}\right)^2\right]} u'$$

$$\frac{du_L}{dt} = (4-d)u_L - \frac{3A\left(e^t, d\right)}{e^{(d-2)t}\left[r_L + \left(\frac{\pi}{e^t a}\right)^2\right]^2} u'^2$$

These equations are valid for all L. In particular, they should hold for $L \simeq 1_+$ or $t \simeq 0_+$. In which case, we have

$$\frac{dr_L}{dt} = 2r_L + \frac{A}{\left[r_L + \left(\frac{\pi}{a}\right)^2\right]} u'$$

$$\frac{du_L}{dt} = (4-d)u_L - \frac{3A}{\left[r_L + \left(\frac{\pi}{a}\right)^2\right]^2} u'^2$$

where $A = A(1,d)$. At a fixed point, we have

$$\frac{dr^*}{dt} = \frac{du^*}{dt} = 0$$

so that

$$2r^* + \frac{A}{r^* + B} u^* = 0 \qquad (1)$$

$$\varepsilon u^* - \frac{3A}{\left(r^* + B\right)^2} u^{*2} = 0 \qquad (2)$$

where $\varepsilon = 4 - d$ and $B = \left(\frac{\pi}{a}\right)^2$. Note that the $L \simeq 1_+$ approximation corresponds to an infinitesimal Kadanoff transformation. Eqs(1,2) are then necessary conditions for fixed points with L arbitrarily large. They may not be sufficient conditions.

The solutions to (2) are,

$$u^* = 0 \qquad \text{or} \qquad u^* = \frac{(r^* + B)^2}{3A}\,\varepsilon$$

Putting these into (1) gives

1. $u^* = 0$ and $r^* = 0$.

2. $u^* = \dfrac{(r^* + B)^2}{3A}\,\varepsilon$ and

$$r^* = -\frac{1}{2}\cdot\frac{A}{r^* + B}\,\frac{(r^* + B)^2}{3A}\,\varepsilon \quad = -\frac{1}{6}(r^* + B)\,\varepsilon \quad = -\frac{B\varepsilon}{6+\varepsilon}$$

so that

$$u^* = \frac{B^2}{3A}\left(-\frac{\varepsilon}{6+\varepsilon} + 1\right)^2 \varepsilon \quad = \frac{B^2}{3A}\left(\frac{6}{6+\varepsilon}\right)^2 \varepsilon$$

The stability matrix is

$$\mathcal{A} = \begin{pmatrix} \dfrac{\partial \dot{r}_L}{\partial r_L} & \dfrac{\partial \dot{r}_L}{\partial u_L} \\ \dfrac{\partial \dot{u}_L}{\partial r_L} & \dfrac{\partial \dot{u}_L}{\partial u_L} \end{pmatrix}_{r_L = r^*,\, u_L = u^*}$$

Now,

$$\frac{\partial \dot{r}_L}{\partial r_L} = 2 - \frac{A}{(r^* + B)^2}\,u^* \quad \simeq 2 - \frac{A}{B^2}\,u^*$$

$$\frac{\partial \dot{r}_L}{\partial u_L} = \frac{A}{r^* + B} \quad \simeq \frac{A}{B}\left(1 - \frac{r^*}{B}\right)$$

$$\frac{\partial \dot{u}_L}{\partial r_L} = \frac{6A}{(r^* + B)^3}\,u^{*2} \simeq 0$$

$$\frac{\partial \dot{u}_L}{\partial u_L} = \varepsilon - \frac{6A}{(r^* + B)^2}\,u^* \quad \simeq \varepsilon - \frac{6A}{B^2}\,u^*$$

so that

$$\mathcal{A} = \begin{pmatrix} 2 - \frac{A}{B^2} u^* & \frac{A}{B}\left(1 - \frac{r^*}{B}\right) \\ 0 & \varepsilon - \frac{6A}{B^2} u^* \end{pmatrix}$$

$\Rightarrow$

$$\lambda_r = 2 - \frac{A}{B^2} u^*$$

$$\lambda_u = \varepsilon - \frac{6A}{B^2} u^*$$

For $\left(r^*, u^*\right) = (0,0)$ [Gaussian fixed point], we have

$$\lambda_r = 2 > 0 \qquad \text{relevant}$$

$$\lambda_u = \varepsilon \underset{<}{>} 0 \quad \text{for} \quad d \underset{>}{<} 4 \qquad \begin{matrix} \textit{relevant} \\ \textit{irrelevant} \end{matrix}$$

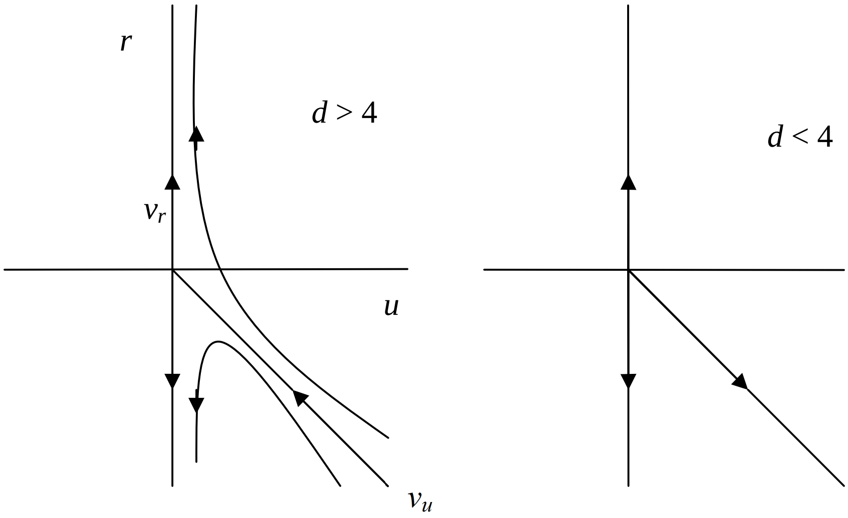
Thus, for $d < 4$, both λ_r and λ_u are positive so that the fixed point $(0,0)$ is unstable. For $d > 4$, we have $\lambda_r > 0$ and $\lambda_u < 0$ so that the fixed point $(0,0)$ is hyperbolic with λ_u corresponding to a physical quantity.

For $\lambda_r = 2$,

$$\begin{pmatrix} 0 & \frac{A}{B} \\ 0 & \varepsilon - 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0 \qquad \Rightarrow \qquad v_r = \begin{pmatrix} c_1 \\ 0 \end{pmatrix} \qquad \text{(right eigenvector)}$$

For $\lambda_u = \varepsilon$,

$$\begin{pmatrix} 2 - \varepsilon & \frac{A}{B} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0 \qquad \Rightarrow \qquad v_u = \begin{pmatrix} c_1 \\ -\frac{B}{A}(2 - \varepsilon) \end{pmatrix}$$



Note that $u \geq 0$.

For $\left(r^*, u^*\right) = \left(-\frac{B\varepsilon}{6+\varepsilon}, \frac{B^2}{3A}\left(\frac{6}{6+\varepsilon}\right)^2 \varepsilon\right)$, we have

$$\lambda_r = 2 - \frac{1}{3}\left(\frac{6}{6+\varepsilon}\right)^2 \varepsilon > 0$$

$$\lambda_u = \varepsilon - 2\left(\frac{6}{6+\varepsilon}\right)^2 \varepsilon \quad = -\varepsilon\left[2\left(\frac{6}{6+\varepsilon}\right)^2 - 1\right] \qquad \begin{matrix} > 0 \\ < 0 \end{matrix} \quad \text{for} \quad d \begin{matrix} > \\ < \end{matrix} 4$$

Hence, only the $d < 4$ case is physical.

Note that both λ_r and λ_u are independent of A and B at the fixed points. This must be so or else the critical exponents to be calculated will be meaningless.

Along the eigencurve of λ_r, we have

$$\delta v_r\left(t\right) = \exp\left(\lambda_r t\right)\delta v_r\left(0\right) \quad = \exp\left(\lambda_r \ln L\right)\delta v_r\left(0\right)$$

For n transformations

$$\delta v_r\left(nt\right) = \exp\left(n\lambda_r t\right)\delta v_r\left(0\right) \quad = \left[\exp\left(\lambda_r \ln L\right)\right]^n \delta v_r\left(0\right)$$

Hence,

$$\lambda_1 = \exp\left(\lambda_r \ln L\right)$$

Assigning r to $t = \dfrac{T - T_C}{T}$, we have

$$p = \frac{\ln \lambda_1}{d \ln L} = \frac{\lambda_r \ln L}{d \ln L} = \frac{\lambda_r}{d}$$

Hence, for $d \ge 4$, we have $(0,0)$ as a fixed point so that

$$\lambda_r = 2 \qquad \Rightarrow \qquad p = \frac{2}{d}$$

$$\alpha = 2 - \frac{1}{p} = 2 - \frac{d}{2} = 2 - \frac{4-\varepsilon}{2} = \frac{\varepsilon}{2}$$

$$\nu = \frac{2-\alpha}{d} = \frac{1}{pd} = \frac{1}{2}$$

To calculate δ, we need to include a term $-B's'(\mathbf{k}=0)$ to H. But the result will be the same as the Gaussian model. Hence,

$$\delta = \frac{d+2}{d-2} = \frac{4-\varepsilon+2}{4-\varepsilon-2} = \frac{6-\varepsilon}{2-\varepsilon}$$

$$\underset{\varepsilon \to 0}{\rightarrow} \frac{6-\varepsilon}{2}\left(1+\frac{\varepsilon}{2}\right) \simeq 3 + \left(-\frac{\varepsilon}{2} + \frac{3}{2}\varepsilon\right) = 3 + \varepsilon$$

For $d < 4$, we have $\left(-\dfrac{B\varepsilon}{6+\varepsilon}, \dfrac{B^2}{3A}\left(\dfrac{6}{6+\varepsilon}\right)^2 \varepsilon\right)$ as fixed point so that

$$\lambda_r = 2 - \frac{1}{3}\left(\frac{6}{6+\varepsilon}\right)^2 \varepsilon \simeq 2 - \frac{\varepsilon}{3} \qquad \text{as } \varepsilon \to 0$$

$$p \simeq \frac{2 - \frac{\varepsilon}{3}}{d}$$

$$\alpha \simeq 2 - \frac{4-\varepsilon}{2 - \frac{\varepsilon}{3}} \simeq 2 - \frac{4-\varepsilon}{2}\left(1+\frac{\varepsilon}{6}\right) \simeq 2 - \left(2 - \frac{\varepsilon}{2}\right) - \frac{\varepsilon}{3} = \frac{\varepsilon}{6}$$

$$\nu = \frac{1}{2 - \frac{\varepsilon}{3}} \simeq \frac{1}{2}\left(1+\frac{\varepsilon}{6}\right) \simeq \frac{1}{2} + \frac{\varepsilon}{12}$$

$$\delta \simeq 3 + \varepsilon \qquad \text{(same as the (0,0) fixed point)}$$

For $d = 3$, we have $\varepsilon = 1$ so that

$$\alpha = \frac{1}{6} \simeq 0.17$$

$$\nu = \frac{1}{2} + \frac{1}{12} \simeq 0.58$$

$$\delta = 4$$

Note that the $\varepsilon \to 0$ result is applicable as long as $\left(\frac{6}{6+\varepsilon}\right) \simeq 1$.

For $d = 4$, we have $\varepsilon = 0$ so that

$$\alpha = 0$$

$$\nu = \frac{1}{2} \qquad \text{(same as mean field)}$$

$$\delta = 3$$

S8.B. Exact Solution of the Two-Dimensional Ising Model

S8.B.1. Partition Function

S8.B.2. Antisymmetric Matrices and Dimer Graphs

S8.B.3. Closed Graphs and Mixed Dimer Graphs

S8.B.4. Partition Function for Infinite Planar Lattice