3.G. Ginzburg-Landau Theory

A phase transition can be transformed into an order-disorder transition by the introduction of an order parameter η so that the ordered (disordered, or less ordered) phase is characterized by $\eta \neq 0$ ($\eta = 0$). In this sense, a phase transition is said to be continuous (discontinuous) if η is continuous (discontinuous) at the transition point.

As an example, for the familiar PVT system, the order parameter of phase α is related to the density ρ by

$$\eta_\alpha = \rho_\alpha - \rho_{\text{HT}}$$

where ρ_{HT} is the density of the higher temperature phase. Hence, η is discontinuous for all transitions except for the critical point on the vaporization curve.

Another aspect of great theoretical importance is whether symmetry of the system is lowered (or broken) by the phase transition. The behavior of η and symmetry is summarized in the following table.

	1 st order	continuous
η	discontinuos	continuous
broken symmetry	perhaps	always

Consider a YXT system with an order parameter η coupled to an external force f. The Ginzburg-Landau theory assumes the Gibbs free energy ϕ to be an analytic function of the order parameter η. Near the critical point where $\eta = 0$, its Taylor expansion up to $O(\eta^4)$ takes the form

$$\phi(T, Y, \eta) = \phi_0(T, Y) + \alpha_1 \eta + \frac{1}{2!} \alpha_2 \eta^2 + \frac{1}{3!} \alpha_3 \eta^3 + \frac{1}{4!} \alpha_4 \eta^4 \tag{3.63a}$$

where the coefficients α_j are all real functions of the intensive variables (T, Y) and

$$\phi_0(T, Y) = \phi(T, Y, 0)$$

In the presence of the external force f, we switch to the Legendre transform

$$\psi(T, Y, f) = \phi(T, Y, \eta) - f \eta \tag{3.63b}$$

Comment: By definition, ϕ, as a function of (T, Y), is non-analytic at the critical point. The introduction of η thus "regularizes" the problem so that the usual mathematical tools apply. All non-analyticity are shifted to the coefficients $\alpha_j(T, Y)$.

In the study of various systems, η in (3.63a) was found to be a complex scalar (for superfluids), a vector (for magnets), and a tensor (for liquid crystals). Since ϕ is a scalar, η must enter in a suitable tensorial combination that gives a scalar. In such cases, the order parameter is taken as the magnitude $|\eta| \geq 0$.

Now, in order to confine the system to the vicinity of $\eta = 0$, we set

$$\alpha_4 > 0 \tag{3.66}$$

so that [see Fig.3.20]

$$\phi \to +\infty \quad \text{as} \quad |\eta| \to \infty \tag{3.66a}$$

Consider first the case $f = 0$.

Let T_X be the transition temperature. If $\eta = 0$ is to be the high temperature phase, then the global minimum of ϕ must be

(1) at $\eta = 0$ for $T > T_X$,

and

(2) at some $\eta \neq 0$ for $T < T_X$.

Taking the derivatives of (3.63) gives

$$\frac{\partial \phi}{\partial \eta} = \alpha_1 + \alpha_2\, \eta + \frac{1}{2}\alpha_3\, \eta^2 + \frac{1}{3!}\alpha_4\, \eta^3 \qquad \text{(a)}$$

$$\frac{\partial^2 \phi}{\partial \eta^2} = \alpha_2 + \alpha_3\, \eta + \frac{1}{2}\alpha_4\, \eta^2 \qquad \text{(b)}$$

The extrema of ϕ are roots of the cubic equation

$$\frac{\partial \phi}{\partial \eta} = \alpha_1 + \alpha_2\, \eta + \frac{1}{2}\alpha_3\, \eta^2 + \frac{1}{3!}\alpha_4\, \eta^3 = 0 \qquad (3.65)$$

In order for $\eta = 0$ to be a root, we must have

$$\alpha_1 = 0$$

so that (3.63a) simplifies to

$$\phi(T, Y, \eta) = \phi_0(T, Y) + \frac{1}{2!}\alpha_2\, \eta^2 + \frac{1}{3!}\alpha_3\, \eta^3 + \frac{1}{4!}\alpha_4\, \eta^4 \qquad (3.63)$$

and (3.65) becomes

$$\eta\left(\alpha_2 + \frac{1}{2}\alpha_3\, \eta + \frac{1}{3!}\alpha_4\, \eta^2\right) = 0 \qquad \text{(c)}$$

with roots

$$\eta_0 = 0 \qquad \text{(d)}$$

and

$$\eta_\pm = \frac{3}{2\,\alpha_4}\left(-\alpha_3 \pm \sqrt{\alpha_3^2 - \frac{8}{3}\alpha_2\,\alpha_4}\right)$$

$$= \frac{3}{2}\left(-\tilde{\alpha}_3 \pm \sqrt{-\Delta}\right) \qquad \text{(e)}$$

where we have introduced the reduced quantities

$$\tilde{x} \equiv \frac{x}{\alpha_4} \qquad [\,\alpha_4 > 0\,]$$

and

$$\Delta = \frac{8}{3}\tilde{\alpha}_2 - \tilde{\alpha}_3^2 \qquad \text{(f)}$$

is the discriminant that determines the character of $\eta_\pm$.

For $\Delta > 0$, $\eta_\pm$ are complex conjugates to each other.

For $\Delta \leq 0$, $\eta_\pm$ are real, with $\eta_+ = \eta_-$ if $\Delta = 0$.

Assuming η to be real, the constraint (3.66a) implies,

For $\Delta > 0$, η_0 is the only extrema, which must be a minimum.

For $\Delta \leq 0$, there are 2 minima and 1 maximum among η_0 & $\eta_\pm$.

This suggests that Δ can be used as a measure of the distance from the transition point, designated by $\Delta = \Delta_X$.

At the extrema η_0 & $\eta_\pm$, we have

$$\tilde{\phi}(T, Y, \eta_0) = \tilde{\phi}_0(T, Y) \qquad \text{(g1)}$$

$$\left.\frac{\partial^2 \tilde{\phi}}{\partial \eta^2}\right|_{\eta=0} = \tilde{\alpha}_2 = \frac{3}{8}\left(\Delta + \tilde{\alpha}_3^2\right) \qquad \text{(g2)}$$

and

$$\tilde{\phi}(T, Y, \eta_\pm) = \tilde{\phi}_0(T, Y) + \eta_\pm^2\left(\frac{1}{2}\tilde{\alpha}_2 + \frac{1}{3!}\tilde{\alpha}_3\,\eta_\pm + \frac{1}{4!}\eta_\pm^2\right)$$

$$= \tilde{\phi}_0(T, Y) + \frac{1}{4}\eta_\pm^2\left(\tilde{\alpha}_2 + \frac{1}{6}\tilde{\alpha}_3\,\eta_\pm\right) \qquad \text{[(c) used.]} \qquad \text{(h1)}$$

Similarly, using (c) to eliminate η^2 in (b) gives

$$\left.\frac{\partial^2 \tilde{\phi}}{\partial \eta^2}\right|_{\eta=\eta_\pm} = -2\,\tilde{\alpha}_2 - \frac{3}{2}\tilde{\alpha}_3\,\eta_\pm$$

$$= -2\,\tilde{\alpha}_2 - \frac{9}{4}\tilde{\alpha}_3\left(-\tilde{\alpha}_3 \pm \sqrt{-\Delta}\right) \qquad \text{[(e) used.]}$$

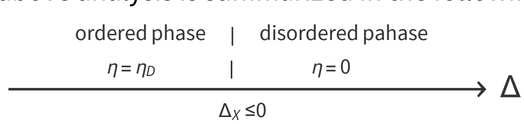
$$= -\frac{3}{4}\left(\Delta - 2\,\tilde{\alpha}_3^2 \pm 3\,\tilde{\alpha}_3\sqrt{-\Delta}\right) \qquad \text{(h2)}$$

For $\Delta > 0$, (g2) gives

$$\left.\frac{\partial^2 \tilde{\phi}}{\partial \eta^2}\right|_{\eta=0} = \frac{3}{8}\left(\Delta + \tilde{\alpha}_3^2\right) > 0$$

so that η_0 is the only minimum and the disordered is the stable phase. Hence, $\Delta_X \le 0$.

The above analysis is summarized in the following diagram.



For $\Delta < 0$, there are two scenarios

(A) η_0 remains a minimum.

(B) η_0 becomes the maximum.

In either case, the ordered phase is the stable one only if

$$\tilde{\phi}(T, Y, \eta_D) < \tilde{\phi}(T, Y, \eta_0) = \tilde{\phi}_0(T, Y)$$

where η_D is the $\eta_\pm$ that gives the lower ϕ.

Let $\eta_D = \eta_X$ at the transition point, then

$$\tilde{\phi}\,(T_X, Y_X, \eta_X) = \tilde{\phi}_0\,(T_X, Y_X)$$

where the subscript X denotes values at the transition point. (h1) then gives

$$\eta_X^2\left(\tilde{\alpha}_{2X} + \frac{1}{6}\tilde{\alpha}_{3X}\,\eta_X\right) = 0$$

so that either

$$\eta_X = \frac{3}{2}\left(-\tilde{\alpha}_{3X} \pm \sqrt{-\Delta_X}\right) = 0 \qquad \text{[transition is continuous]} \qquad \text{(i1)}$$

or

$$\eta_X = \frac{3}{2}\left(-\tilde{\alpha}_{3X} \pm \sqrt{-\Delta_X}\right) = -\frac{6\,\tilde{\alpha}_{2X}}{\tilde{\alpha}_{3X}} \qquad \text{[transition is discontinuous if } \tilde{\alpha}_{2X} \neq 0 \text{]} \qquad \text{(i2)}$$

For the continuous transition (i1),

$$\Delta_X = -\tilde{\alpha}_{3X}^2 \qquad \text{(j1)}$$

which, according to (f), also implies

$$\Delta_X = \frac{8}{3}\tilde{\alpha}_{2X} - \tilde{\alpha}_{3X}^2 = -\tilde{\alpha}_{3X}^2 \qquad \to \qquad \tilde{\alpha}_{2X} = 0 \qquad \text{(j2)}$$

For the 1st order transition, solving (i2) for Δ_X gives

$$\Delta_X = -\frac{1}{\tilde{\alpha}_{3X}^2}\left(\tilde{\alpha}_{3X}^2 - 4\,\tilde{\alpha}_{2X}\right)^2 \qquad \text{(j3)}$$

$$= \frac{8}{3}\tilde{\alpha}_{2X} - \tilde{\alpha}_{3X}^2 \qquad \text{[(f) used.]}$$

$$\to \quad \tilde{\alpha}_{2X}\left(\frac{1}{3} - \frac{\tilde{\alpha}_{2X}}{\tilde{\alpha}_{3X}^2}\right) = 0$$

Since $\tilde{\alpha}_{2X} = 0$ makes the transition continuous, we have

$$\tilde{\alpha}_{2X} = \frac{1}{3}\tilde{\alpha}_{3X}^2 \qquad \text{(j4)}$$

so that (j3) becomes

$$\Delta_X = -\frac{1}{9}\tilde{\alpha}_{3X}^2 \qquad \text{(j5)}$$

To simplify the analysis, we shall assume α_3 to be a constant and use (j5) to set

$$\tilde{\alpha}_3 = -3\sqrt{-\Delta_X} \qquad \text{(k1)}$$

where the negative root is chosen to ensure $\eta_X > 0$ [see (i2)]. (f) then gives

$$\tilde{\alpha}_2 = \frac{3}{8}\left(\Delta + \tilde{\alpha}_3^2\right) = \frac{3}{8}\left(\Delta - 9\,\Delta_X\right) \qquad \text{(k2)}$$

Note that

$$\Delta_X = 0 \quad \to \quad \tilde{\alpha}_3 = 0$$

$$\to \quad \eta_X = 0 \qquad \text{[(i1) used.]} \qquad \text{(k3)}$$

so that the transition is continuous.

Putting (k1) & (k2) into (3.65a) gives

$$\tilde{\phi}(T, Y, \eta) \cong \phi_0(T, Y) + \frac{3}{16}\left(\Delta - 9\,\Delta_X\right)\eta^2 - \frac{1}{2}\sqrt{-\Delta_X}\,\eta^3 + \frac{1}{4!}\eta^4 \qquad \text{(k4)}$$

The system is therefore characterized by two parameters Δ & Δ_X.

The parameter Δ indicates the equilibrium (stable) state of the system with

- $\Delta > 0 \quad \to \quad \tilde{\phi}$ has 1 minimum
- $\Delta < 0 \quad \to \quad \tilde{\phi}$ has 2 minima
- $\Delta > \Delta_X \quad \to$ stable state is the disordered state with $\eta = 0$.
- $\Delta = \Delta_X \quad \to$ system at transition point (disordered & ordered states coexist).
- $\Delta < \Delta_X \quad \to$ stable state is the ordered state with $\eta \neq 0$.

The magnitude of the parameter Δ_X indicates the discontinuity of η at the transition point.

$\Delta_X = 0$ → transition is continuous

$\Delta_X < 0$ → transition is discontinuous (1st order) with discontinuity

$$\eta_X = -2\,\tilde{\alpha}_3 = 6\sqrt{-\Delta_X} \qquad \text{[(i2), (j4) \& (k1) used.]}$$

3.G.1. Continuous Phase Transitions

Following Reichl, we shall consider the case $\alpha_3 = 0$. For the ease of reference, we shall gather the salient equations in the following.

$$\tilde{\phi}(T, Y, \eta) = \tilde{\phi}_0(T, Y) + \frac{1}{2}\tilde{\alpha}_2\,\eta^2 + \frac{1}{4!}\eta^4 \qquad \text{[(3.65a) used.]} \qquad \text{(A1)}$$

$$\eta_\pm = \pm\frac{3}{2}\sqrt{-\Delta} \qquad \text{[(e) used.]} \qquad \text{(A2)}$$

$$\Delta = \frac{8}{3}\tilde{\alpha}_2 \qquad \text{[(f) used.]} \qquad \text{(A3)}$$

$$\tilde{\phi}(T, Y, \eta) = \tilde{\phi}_0(T, Y) + \frac{3}{16}\Delta\,\eta^2 + \frac{1}{4!}\eta^4 \qquad \text{(A4)}$$

$$\tilde{\phi}(T, Y, \eta_\pm) = \tilde{\phi}_0(T, Y) + \frac{1}{4}\tilde{\alpha}_2\,\eta_\pm^2 \qquad \text{[(h1) used.]} \qquad \text{(A5)}$$

$$= \tilde{\phi}_0(T, Y) - \frac{27}{128}\Delta^2 \qquad \text{(A6)}$$

$$\left.\frac{\partial^2\tilde{\phi}}{\partial\eta^2}\right|_{\eta=0} = \tilde{\alpha}_2 = \frac{3}{8}\Delta \qquad \text{[(g2) used.]} \qquad \text{(A7)}$$

$$\left.\frac{\partial^2\tilde{\phi}}{\partial\eta^2}\right|_{\eta=\eta_\pm} = -\frac{3}{4}\Delta \qquad \text{[(h2) used.]} \qquad \text{(A8)}$$

$$\eta_X = 0 \quad \rightarrow \quad \Delta_X = 0 \qquad \text{(A9)}$$

Thus, the stable Gibbs energy is given by

$$\tilde{\phi}(T, Y) = \begin{cases} \tilde{\phi}(T, Y, \eta_0) & \text{for } \Delta \ge \Delta_X \\ \tilde{\phi}(T, Y, \eta_\pm) & \text{for } \Delta \le \Delta_X \end{cases}$$

$$= \begin{cases} \tilde{\phi}_0(T, Y) & \text{for } \Delta \ge 0 \\ \tilde{\phi}_0(T, Y) - \frac{27}{128}\Delta^2 & \text{for } \Delta \le 0 \end{cases} \qquad \text{(3.70)}$$

The coexistence curve $Y(T)$, i.e., the λ-line, satisfies

$$\Delta(T, Y) = 0$$

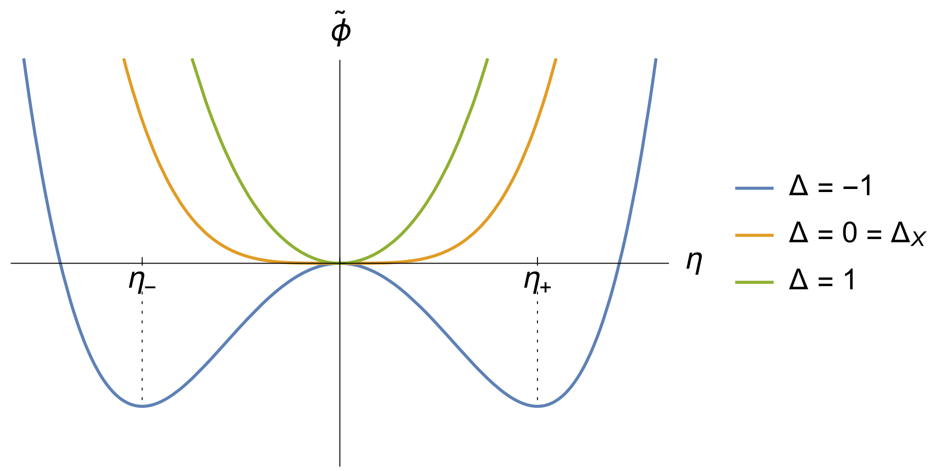


Fig.3.20. Plot of $\tilde{\phi}(\eta) = \frac{3}{16}\Delta\,\eta^2 + \frac{1}{4!}\eta^4$ for $\Delta = -1, 0, 1$.

Fig.3.20 shows the plot of $\tilde{\phi}(\eta)$ for various Δ with $\tilde{\phi}_0 = 0$. The bottom of the curve at the transition point (in orange) is flat because every derivative at $\eta = 0$ vanishes:

$$\left.\frac{\partial^m \tilde{\phi}}{\partial\,\eta^m}\right|_{\eta=0} = 0 \qquad \forall\ m = 1, 2, 3, \ldots$$

The molar heat capacity at constant Y is given by

$$c_Y = -T\left(\frac{\partial^2 \phi}{\partial\,T^2}\right)_Y \tag{3.71}$$

$$= \begin{cases} c_{Y0} & \text{for } \Delta > \Delta_X \\ c_{Y0} + \frac{27}{128} T\left[\frac{\partial^2(\alpha_4\,\Delta^2)}{\partial\,T^2}\right]_Y & \text{for } \Delta < \Delta_X \end{cases} \qquad \text{[(3.70) used.]} \tag{3.71a}$$

where

$$c_{Y0} = -T\left(\frac{\partial^2 \phi_0}{\partial\,T^2}\right)_Y \tag{3.71b}$$

is the molar heat capacity when $\eta = 0$.

The discontinuity at Δ_X is

$$\Delta\,c_Y = c_Y(\Delta_X - \delta) - c_Y(\Delta_X + \delta) \qquad \delta \to 0_+$$

$$= \frac{27}{128} T \left.\frac{\partial^2(\alpha_4\,\Delta^2)}{\partial\,T^2}\right|_{T=T_X}$$

$$= \frac{3}{2} T \left.\frac{\partial^2}{\partial\,T^2}\left(\frac{\alpha_2^2}{\alpha_4}\right)\right|_{T=T_X} \tag{3.72a}$$

In order to get a “feel” of the problem, we consider transition between para- and ferro-magnetism. To keep things simple, we shall assume

1. paramagnetism is characterized by

$$\boldsymbol{M} = 0 \text{ for } \boldsymbol{H} = 0 \qquad \& \qquad \chi = \frac{C}{T - T_C} \qquad \text{[Curie-Weiss law.]}$$

where $\boldsymbol{M}$ is the magnetization, $\boldsymbol{H}$ the magnetic field, χ the magnetic susceptibility,

C a material constant and T_C the Curies' temperature.

2. ferromagnetism is characterized by

$$\boldsymbol{M} \neq 0 \text{ for } \boldsymbol{H} \neq 0$$

and that the whole system is in a single magnetic domain ($\boldsymbol{M}$ is the same everywhere).

Choosing $|\boldsymbol{M}|$ as the order parameter, we set

$$\tilde{\psi}(T, \boldsymbol{H}) = \tilde{\psi}_0(T) + \frac{1}{2} \tilde{\alpha}_2 \boldsymbol{M} \cdot \boldsymbol{M} + \frac{1}{4!} (\boldsymbol{M} \cdot \boldsymbol{M})^2 - \boldsymbol{M} \cdot \tilde{\boldsymbol{H}} \tag{3.78}$$

$$= \tilde{\psi}_0(T) + \frac{1}{2} \tilde{\alpha}_2 M^2 + \frac{1}{4!} M^4 - M \tilde{H} \qquad [\, M = |\boldsymbol{M}| = \sqrt{\boldsymbol{M} \cdot \boldsymbol{M}} \,] \tag{3.78a}$$

where $\tilde{\boldsymbol{H}} = \boldsymbol{H} / \alpha_4$ is the reduced magnetic field and we have assumed $\boldsymbol{M} // \boldsymbol{H}$. Otherwise, the order parameter must be chosen as the vector $\boldsymbol{M}$ itself.

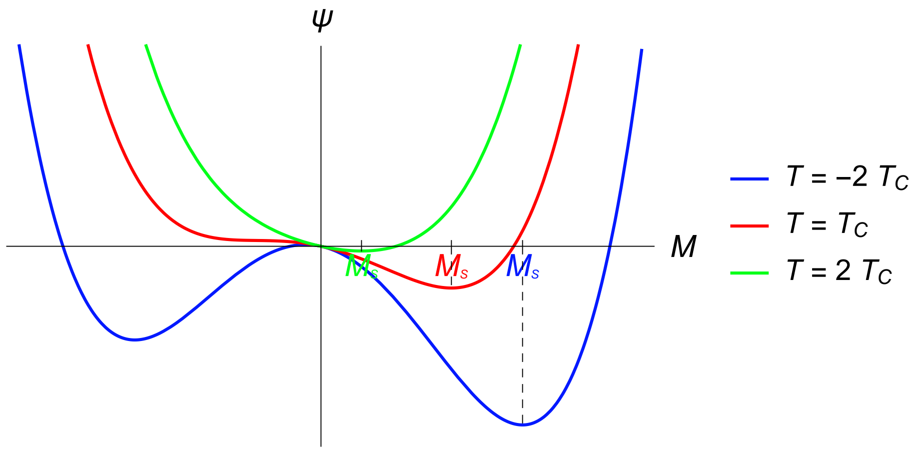


Fig.3.22. Plot of $\psi(M)$ with $\alpha_2 \propto (T - T_C)$ for various T. M_S is the global minimum.

For $H = 0$, (3.78a) is just (A1) with $\eta = M$. Therefore, the relations (A2) to (A9), as well as (3.72a) apply. In particular, since the actual physical state must have $M \geq 0$,

$$M = \begin{cases} 0 & \text{for } \Delta > 0 \\ \frac{3}{2}\sqrt{-\Delta} & \text{for } \Delta < 0 \end{cases} \qquad [\text{ (A2) used. }] \tag{3.78b}$$

For $H \neq 0$, we have

$$\frac{\partial \tilde{\psi}}{\partial M} = \tilde{\alpha}_2 M + \frac{1}{3!} M^3 - \tilde{H} \tag{3.75a}$$

$$\frac{\partial^2 \tilde{\psi}}{\partial M^2} = \tilde{\alpha}_2 + \frac{1}{2} M^2 \tag{3.75b}$$

At the equilibrium value M_s, ψ is a global minimum so that

$$\tilde{\alpha}_2 M_s + \frac{1}{3!} M_s^3 - \tilde{H} = 0 \qquad \& \qquad \tilde{\alpha}_2 + \frac{1}{2} M_s^2 > 0 \tag{3.75c}$$

Thus,

$$M_s \neq 0 \quad \text{if} \quad H \neq 0$$

If the H-dependence of α_2 & α_4 can be neglected, $\frac{\partial}{\partial H}(3.75\,c)$ gives

$$\tilde{\alpha}_2 \frac{\partial M_s}{\partial H} + \frac{1}{2} M_s^2 \frac{\partial M_s}{\partial H} - \frac{1}{\alpha_4} = 0$$

so that the magnetic susceptibility χ is

$$\chi \equiv \lim_{H\to 0} \left(\frac{\partial M_s}{\partial H}\right)_T$$

$$= \lim_{H\to 0} \frac{1}{\alpha_4\left(\tilde{\alpha}_2 + \frac{1}{2} M_s^2\right)} \qquad (3.76)$$

$$= \begin{cases} \dfrac{1}{\alpha_2} & \text{for } \Delta > 0 \\ \dfrac{1}{\alpha_4\left(\tilde{\alpha}_2 - \frac{9}{8}\Delta\right)} = -\dfrac{1}{2\,\alpha_2} & \text{for } \Delta < 0 \end{cases} \qquad \text{[(3.78b) \& (A3) used.]} \qquad (3.77a)$$

Comparing with the Curie-Weiss law, we have

$$\alpha_2 = \frac{1}{C}(T - T_C) \qquad (3.67)$$

so that (3.77a) becomes

$$\chi = \begin{cases} \dfrac{C}{T - T_C} & \text{for } \Delta > 0 \\ -\dfrac{C}{2\,(T - T_C)} & \text{for } \Delta < 0 \end{cases} \qquad (3.77)$$

Therefore, $\Delta_X = 0$ [see (A9)] gives

$$T_X = T_C \qquad (3.77b)$$

as expected. Putting (3.67) into (A3) gives

$$\Delta = \frac{8}{3} \frac{1}{\alpha_4\,C}(T - T_C) \qquad (3.77c)$$

which allows us to write (3.78b) as

$$M_s = \begin{cases} 0 & \text{for } T > T_C \\ \sqrt{\dfrac{6}{\alpha_4\,C}(T_C - T)} & \text{for } T < T_C \end{cases} \qquad (3.77d)$$

Reminder: our α_4 is $4!$ times Reichl's and $C = \frac{1}{2}\alpha_0$.

Finally, if we neglect the T-dependence of α_4, (3.71a) gives

$$c_H = \begin{cases} c_{H0} & \text{for } T > T_C \\ c_{H0} + \dfrac{3}{2\,\alpha_4\,C^2} T \dfrac{\partial^2}{\partial T^2}(T - T_C)^2 & \text{for } T < T_C \end{cases} \qquad \text{[(3.77c) used.]}$$

$$= \begin{cases} c_{H0} & \text{for } T > T_C \\ c_{H0} + \dfrac{3}{\alpha_4\,C^2} T & \text{for } T < T_C \end{cases} \qquad (3.77e)$$

so that the discontinuity at the transition point is

$$\Delta c_H = \frac{3}{\alpha_4 C^2} T_C \tag{3.72}$$

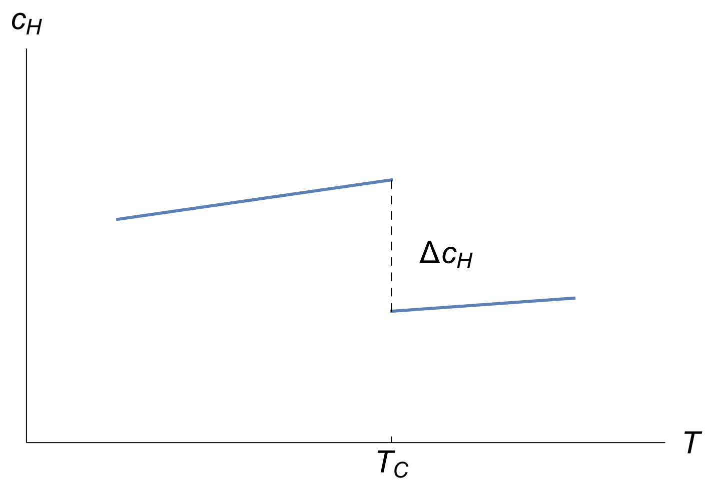


Fig.3.21. $c_H(T)$ assuming $c_{H0} \propto T$.

Another example is the transition between normal & superfluid in liquid He^4 with

$$\begin{aligned}\tilde{\phi}(T, P, \Psi) &= \tilde{\phi}_0(T, P) + \frac{1}{2}\tilde{\alpha}_2 \Psi^* \Psi + \frac{1}{4!}(\Psi^* \Psi)^2 \\ &= \tilde{\phi}_0(T, P) + \frac{1}{2}\tilde{\alpha}_2 \left|\Psi\right|^2 + \frac{1}{4!}\left|\Psi\right|^4\end{aligned} \tag{3.73}$$

where Ψ is the (complex) wave function of the condensation (superfluid component). The 3rd order term is absence since the expansion is actually for powers of $\Psi^* \Psi$. The order parameter for (3.73) is obviously $\eta = |\Psi|$. However, for the general problem that includes superflows, $\tilde{\phi}$ must be written in terms of $\Psi = e^{i\theta}\left|\Psi\right|$ so that $|\eta|$ is the order parameter.

Of interest is the discontinuity in c_P at the transition point [see (3.72a)]

$$\Delta c_P = \frac{3}{2} T \frac{\partial^2}{\partial T^2}\left(\frac{\alpha_2^2}{\alpha_4}\right)\Bigg|_{T=T_X}$$

The λ-shape of $c_P(T)$ shown in Fig.3.16 can be simulated if

$$\alpha_2 \propto (T_X - T)^p$$

where $p < 0$ so that c_P is singular as $T \to T_X$.

3.G.2. 1st Order Transitions

Since the situation was already fully analyzed, we shall conclude with a plot of $\tilde{\phi}(\eta)$ for various Δ's.

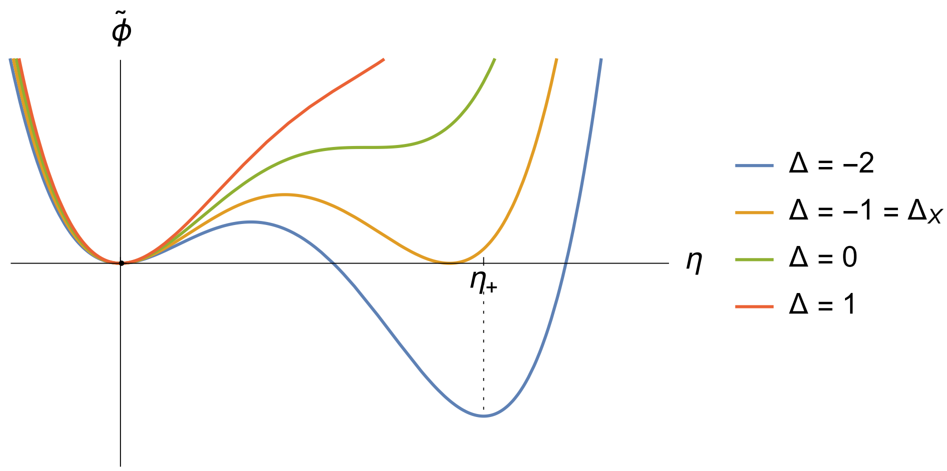


Fig.3.24. Plot of $\tilde{\phi}(\eta)$ for various Δ's with $\phi_0 = 0$ & $\Delta_X = -1$.
The minimum of curve $\Delta = -2$ occurs at η_+.

Code

```
ϕ[η_, Δ_] := (3 Δ)/16 η^2 + 1/(4 !) η^4
```

```
In[∘]:= ηx = η /. Solve[∂_η ϕ[η, -1] == 0, η]
```

```
In[∘]:= (* Fig.3.20 *)
ϕL = Table[ϕ[η, Δ], {Δ, -1, 1, 1}];
Plot[ϕL, {η, -10, 10},
     PlotRange → {All, .3 {-1, 1}},
     AxesLabel → {"η", "ϕ̃"},
 Ticks → {{0, {ηx⟦1⟧, "η_"}, {ηx⟦3⟧, "η₊"}}, {0}},
 PlotLegends → {"Δ = -1", "Δ = 0 = Δ_X", "Δ = 1"},
 Epilog → {Dotted, Line[{{ηx⟦1⟧, 0}, {ηx⟦1⟧, ϕ[ηx⟦1⟧, -1]}}],
           Line[{{ηx⟦3⟧, 0}, {ηx⟦3⟧, ϕ[ηx⟦3⟧, -1]}}]}
]
```

$$\frac{\partial \tilde{\psi}}{\partial \eta} = \tilde{\alpha}_2\, \eta + \frac{1}{3!} \eta^3 - \tilde{f}$$

```
In[∘]:= Solve[α2 η + 1/6 η^3 - f == 0, η]
```

$$= \tilde{\psi}_0(T) + \frac{1}{2} \tilde{\alpha}_2\, M^2 + \frac{1}{4!} M^4 - M\, \tilde{H}$$

```
In[∘]:= ψΔ[M_, Δ_, H_] := (3 Δ)/16 M^2 + 1/(4 !) M^4 - M H
```

```
In[∘]:= H = 1;
Ms = Table[Max[M /. NSolve[∂_M ψΔ[M, Δ, H] == 0, M, Reals]], {Δ, -4, 4, 4}]
```

In[∘]:=
```
Clear[Tc];
ψ[M_, T_, H_] := 1/2 (T - Tc) M^2 + 1/4! M^4 - M H
```

In[∘]:=
```
Tc = 1; H = 1;
Ms = Table[Max[M /. NSolve[∂_M ψ[M, T, H] == 0, M, Reals]], {T, -2 Tc, 2 Tc, 2 Tc}]

(* Fig.3.22 *)
ψL = Table[ψ[M, T, H], {T, -2 Tc, 2 Tc, 2 Tc}];
color = Table[Hue[.3 + .35 i], {i, 3}];
Plot[ψL, {M, -7, 7},
     PlotRange → {All, {-20, 20}},
     PlotStyle → color,
     AxesLabel → {"M", "ψ"},
     Ticks → {Table[{Ms[[i]], Style["M_S", Hue[.3 + .35 i]]}, {i, 3}], None},
     PlotLegends → {"T = -2 T_C", "T = T_C" , "T = 2 T_C"},     Epilog →
  {Dashed, Table[Line[{{Ms[[i]], 0}, {Ms[[i]], ψ[Ms[[i]], -2 Tc + 2 (i - 1) Tc, H]}}], {i, 3}]}
]
```

In[∘]:=
```
cH[T_, n_] := { T^n       T ≥ Tc
              { T^n + T   T < Tc
```

In[∘]:=
```
(* Fig.3.23 *)
Tc = 2;
Plot[cH[T, 1], {T, 1.7, 2.2},
     PlotRange → {{1.6, 2.3}, {0, 6}},
     AxesLabel → {"T", "c_H"},
     Ticks → {{{Tc, "T_C"}}, None},
     Epilog → {Dashed, Line[{{Tc, cH[1.0001 Tc, 1]}, {Tc, cH[.9999 Tc, 1]}}],
       Text["Δc_H", {1.03 Tc, 1.4 cH[1.02 Tc, 1]}]}
]
```

$$\tilde{\phi}(T, Y, \eta) \cong \phi_0(T, Y) + \frac{3}{16}(\Delta - 9\,\Delta_X)\,\eta^2 - \frac{1}{2}\sqrt{-\Delta_X}\;\eta^3 + \frac{1}{4!}\eta^4 \qquad \text{(k4)}$$

In[∘]:=
```
ϕ1[η_, Δ_, Δx_] := 3/16 (Δ - 9 Δx) η^2 - 1/2 Sqrt[-Δx] η^3 + 1/4! η^4
```

In[∘]:=
```
η1x = η /. Solve[∂_η ϕ1[η, -2, -1] == 0, η]
```

In[∘]:=
```
(* Fig.3.24 *)
ϕ1L = Table[ϕ1[η, Δ, -1], {Δ, -2, 1}];
Plot[ϕ1L, {η, -10, 10},
     PlotRange → {{-2, 10}, 10 {-1, 1}},
     AxesLabel → {"η", "ϕ̃"},
 Ticks → {{0, {η1x[[1]], "η_-"}, {η1x[[3]], "η_+"}}, {0}},
 PlotLegends → {"Δ = -2", "Δ = -1 = Δ_X", "Δ = 0", "Δ = 1"},
 Epilog → {Dotted, Line[{{η1x[[1]], 0}, {η1x[[1]], ϕ1[η1x[[1]], -2, -1]}}],
           Line[{{η1x[[3]], 0}, {η1x[[3]], ϕ1[η1x[[3]], -2, -1]}}]}
]
```