4.B. Permutations and Combinations

Finding probabilities is essentially a counting problem based on two fundamental rules.

(a) Rule of addition:

If two mutually exclusive events can occur in m & n different ways separately,

then there are $m + n$ distinct outcomes.

(b) Rule of multiplication:

If, after one event that can occur in m different ways is done,

another event that can occur in n different ways follows,

then there are $m \times n$ distinct outcomes.

We now apply these two rules to find the number of distinct outcomes in two important situations.

(1) Permutation (or arrangement into different orders) of N distinct objects.

(i) The number of (different) permutations (or distinct outcomes) is

$$P_N^N = N!$$

(ii) If only R objects are taken out for permutation, the number of distinct outcomes is

$$P_R^N = \frac{N!}{(N-R)!} \tag{4.1a}$$

Proof:

(i) The problem is equivalent to putting N objects into a line of N boxes.

Each arrangement of distinct order is a distinct permutation (or outcome).

Putting the objects one at a time, we have

N ways to fill the 1st box,

$N - 1$ ways to fill the 2nd box,

$\vdots$

2 ways to fill the $(N-1)^{\text{th}}$ box, and

1 way to fill the N^{th} box.

Applying the rule of multiplication, we have

$$P_N^N = N\,(N-1)\,\ldots\,2\cdot 1 = N!$$

(ii) The problem is equivalent to selecting R of N objects to put into a line of R boxes.

Putting the objects one at a time, we have

N ways to fill the 1st box,

$N - 1$ ways to fill the 2nd box,

$\vdots$

$N - R + 1$ ways to fill the R^{th} box.

Applying the rule of multiplication, we have

$$P_R^N = N\,(N-1)\,\ldots\,(N-R+1) = \frac{N!}{(N-R)!}$$

(2) Combination (or selection without regard to order) of N distinct objects.

(i) The number of (different) combinations (or distinct outcomes) is

$$C_N^N = \frac{P_N^N}{N!} = 1$$

(ii) If only R objects are taken out for combination, the number of distinct outcomes is

$$C_R^N = \frac{P_R^N}{R!} = \frac{N!}{R!\,(N-R)!} \qquad (4.1b)$$

Proof:

In the permutation P_R^N of R objects out of N, we can 1st select R objects out of N without regard of order, and then permute them. Applying the rule of multiplication, we have

$$P_R^N = C_R^N R! \qquad \rightarrow \qquad C_R^N = \frac{P_R^N}{R!} \qquad \text{QED.}$$

Corollary:

Consider a set of N objects composed of n_k identical elements of the k^{th} kind, for $k = 1, \ldots, m$. Thus,

$$N = \sum_{k=1}^{m} n_k = n_1 + \ldots + n_m$$

The number of permutations of these N objects is

$$\frac{N!}{\prod_{k=1}^{m} n_k} = \frac{N!}{n_1! \ldots n_m!} \qquad (4.1c)$$

Ex.4.1.

(a) Find the number of permutations of the letters in the word “ENGINEERING”.

(b) In how many ways are three E’s together?

(c) In how many ways are only two E’s together?

Answer (a)

In the word “ENGINEERING”,

$N = 11 \qquad n_E = 3 \qquad n_G = 2 \qquad n_I = 2 \qquad n_N = 3 \qquad n_R = 1$

By (4.1c), the number of permutations in “ENGINEERING” is

$$\frac{11!}{3! \times 2! \times 2! \times 3! \times 1!} = 277{,}200$$

In[1]:=

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11!/(3!×2!×2!×3!×1!)
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Out[1]= 277 200

Answer (b)

Let $\mathcal{E}$ = EEE. In the word “$\mathcal{E}$NGINRING”,

$N = 9 \qquad n_{\mathcal{E}} = 1 \qquad n_G = 2 \qquad n_I = 2 \qquad n_N = 3 \qquad n_R = 1$

By (4.1c), the number of permutations in “$\mathcal{E}$NGINRING” is

$$\frac{9!}{1! \times 2! \times 2! \times 3! \times 1!} = 15,120$$

which is also the number of ways to keep three E's together.

In[2]:=
$$\frac{9!}{1! \times 2! \times 2! \times 3! \times 1!}$$

Out[2]= 15 120

Answer (c)

From (b), we see that the number of permutations in "ENGINRING" is 15, 120.

Let $\mathcal{E}$ = EE. As shown by the symbol ^ in

*E*N^G^I^N^R^I^N^G^

there are 8 out of 10 possible slots to insert $\mathcal{E}$ into "ENGINRING" without creating the sequence EEE. The same conclusion clearly applies to any permutation of "ENGINRING".

The rule of multiplication then gives the number of ways to keep only two E's together as

$$8 \times 15,120 = 120,960$$

In[3]:= **8 × 15 120**

Out[3]= 120 960