4.D.2. Moments

The n^{th} moment of X is defined as

$$\langle x^n \rangle \equiv \int_{-\infty}^{\infty} dx\, x^n\, P_X(x) \tag{4.13}$$

Important features of $P_X(x)$ are conveyed by a handful of numbers, which include

1. The most probable value

   $$x_p \equiv \max P_X(x)$$

   which gives the location of the peak of $P_X(x)$.

2. The median x_m.

   By definition, a realization of X has an equal chance to be either greater or smaller than x_m, i.e.,

   $$\int_{-\infty}^{x_m} dx\, P_X(x) = \int_{x_m}^{\infty} dx\, P_X(x) = \frac{1}{2} \qquad \text{[Rule of addition \& (4.10a) used.]}$$

   $$= F_X(x_m) \qquad \text{[(4.11) used.]} \tag{4.13a}$$

   Hence, x_m is the "middle" point that separates the left and right halves of $P_X(x)$.

3. The mean (or 1st moment)

   $$\overline{x} = \langle x \rangle = \int_{-\infty}^{\infty} dx\, x\, P_X(x) \tag{4.13b}$$

   which gives the "center of mass" of $P_X(x)$.

4. The variance (or 2nd central moment)

   $$\mathrm{var}(X) \equiv \mu_2 = \left\langle \left(x - \langle x \rangle \right)^2 \right\rangle = \left\langle \left(x^2 - 2\langle x \rangle x + \langle x \rangle^2 \right) \right\rangle = \left\langle x^2 \right\rangle - \langle x \rangle^2 \tag{4.13c}$$

   where the nth central moment (or moment relative to the mean) is defined as

   $$\mu_n = \left\langle \left(x - \langle x \rangle \right)^n \right\rangle \tag{4.13d}$$

   Alternatively, one can use the standard deviation

   $$\sigma_X \equiv \sqrt{\left\langle \left(x - \langle x \rangle \right)^2 \right\rangle} = \sqrt{\left\langle x^2 \right\rangle - \langle x \rangle^2} \tag{4.13e}$$

   Either $\mathrm{var}(X)$ or σ_X gives the "spread" of $P_X(x)$.

5. The 3rd central moment

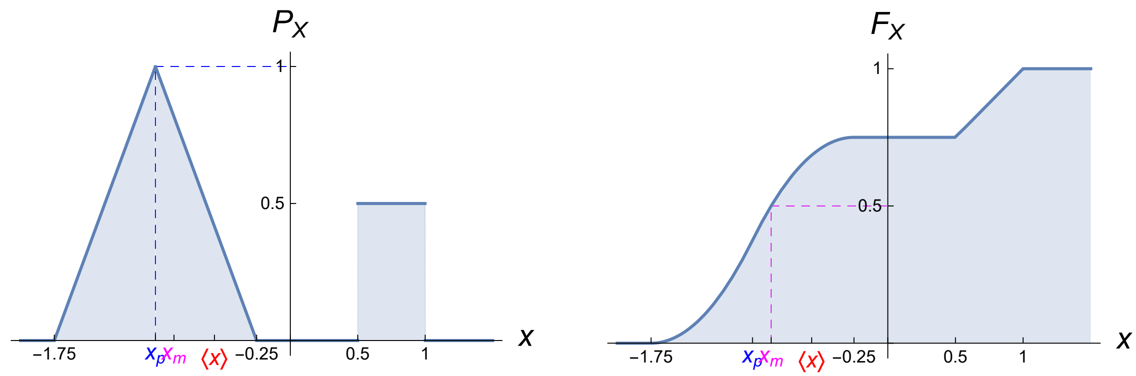
   $$\mu_3 = \left\langle \left(x - \langle x \rangle \right)^3 \right\rangle = \left\langle x^3 - 3\langle x \rangle x^2 + 3\langle x \rangle^2 x - \langle x \rangle^3 \right\rangle = \left\langle x^3 \right\rangle - 3\left\langle x^2 \right\rangle \langle x \rangle + 2\langle x \rangle^3 \tag{4.13f}$$

   which gives the "asymmetry" of $P_X(x)$.

Central moments are used here because they give a better measure of the inherent properties like shape & spread.

Ex.4.5.

Locate $\langle x \rangle$, x_p & x_m for the $P_X(x)$ & $F_X(x)$ shown below.



Answer

Inspection of the graphs gives

$$P_X(x) = \begin{cases} \frac{1}{0.75}(x+1.75) & -1.75 \le x < -1.0 \\ -\frac{1}{0.75}(x+0.25) & -1.0 \le x \le -0.25 \\ 0.5 & 0.5 \le x \le 1.0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Using the *Mathematica* code below, we have

$$\langle x \rangle = -0.5625 \qquad x_p = -1.0 \qquad x_m = -0.8624$$
$$\langle x^2 \rangle = 0.9661 \qquad \langle x^3 \rangle = -0.8438$$
$$\text{var}(X) = 0.6497 \qquad \sigma_X = 0.8061 \qquad \mu_3 = 0.4307 \tag{2}$$

Now, (1) can be approximated by some linear combinations of known probability density functions that satisfy the characteristics given in (2). The advantage of doing so is that the moments can be easily calculated using known formulae of the known density functions. Details will be left as an exercise.

Code

```
In[∘]:= Clear["`Global`*"]

In[∘]:= (* PX(x) *)
PX[x_] := { 1/.75 (x + 1.75)     -1.75 ≤ x < -1.0
            - 1/.75 (x + .25)    -1.0 ≤ x ≤ -.25
            .5                   .5 ≤ x ≤ 1.
            0                    True
```

In[ ]:=
```
FX[x_] = ∫_{-2}^{x} PX[y] dy
```

Integrate: Unable to prove that integration limits {-2, x} are real. Adding assumptions may help.

Out[ ]=
$$\int_{-2}^{x} \left(\begin{cases} 1.33333\,(1.75 + y) & -1.75 \le y < -1. \\ -1.33333\,(0.25 + y) & -1. \le y \le -0.25 \\ 0.5 & 0.5 \le y \le 1. \\ 0 & \text{True} \end{cases} \right) dy$$

In[ ]:=
```
xb = ∫_{-2}^{2} y PX[y] dy
xp = -1.   (* By inspection *)
xm = x /. FindRoot[FX[x] == 1/2, {x, -1}]
```

Out[ ]= -0.5625

Out[ ]= -1.

Integrate: Unable to prove that integration limits {-2, x} are real. Adding assumptions may help.

Out[ ]= -0.862372

In[ ]:=
```
var = ∫_{-2}^{2} (y - xb)^2 PX[y] dy
```

Out[ ]= 0.64974

In[ ]:=
```
σ = √var
```

Out[ ]= 0.806064

In[ ]:=
```
(* moments *)
{∫_{-2}^{2} y^2 PX[y] dy, ∫_{-2}^{2} y^3 PX[y] dy}
```

Out[ ]= {0.966146, -0.84375}

In[ ]:=
```
μ3 = ∫_{-2}^{2} (y - xb)^3 PX[y] dy
```

Out[ ]= 0.430664

In[ ]:=
```
μ3^(1/3)
```

Out[ ]= 0.755173

In[ ]:=
```
tic = {{-1.75, -.25, 0, .5, 1, {xb, "⟨x⟩"}, {xp, "xp"}, {xm, "xm"}}, {0, .5, 1}};
```

In[ ]:=
```
(* PX(x) *)
Plot[PX[x], {x, -2, 1.5},
    AxesLabel → {"x", "PX" },
    Ticks → tic,
    Filling → Bottom,
    Prolog → { Dashed, Blue, Line[{{xp, 0}, {xp, 1}}], Line[{{xp, 1}, {0, 1}}]
  }
]
```

In[ ]:=
```
(* FX(x) *)
Plot[FX[x], {x, -2, 1.5},
    AxesLabel → {"x", "FX" },
    Ticks → tic,
    Filling → Bottom,
    Prolog → {Dashed, Magenta, Line[{{xm, 0}, {xm, .5}}], Line[{{xm, .5}, {0, .5}}]
  }]
```