4.D.3. Characteristic Functions

The 3rd way to specify the probability assignment is by way of the characteristic function defined by

$$f_X(k) \equiv \langle e^{ikx} \rangle \tag{4.15a}$$

$$= \int_{-\infty}^{\infty} dx\, P_X(x)\, e^{ikx} = \text{Fourier transform of } P_X(x). \tag{4.15b}$$

$$= \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} dx\, P_X(x)\, \frac{(ikx)^n}{n!} \qquad [\text{ Taylor expansion of } e^{ikx} \text{ used. }]$$

$$= \sum_{n=0}^{\infty} \frac{(ik)^n \langle x^n \rangle}{n!} \tag{4.15}$$

which specifies the probability assignment in terms of moments.

(4.15) also implies

$$f_X(0) = \int_{-\infty}^{\infty} dx\, P_X(x) = 1 \qquad [\text{ (4.10a) used. }] \tag{4.15c}$$

$$f_X(-k) = \int_{-\infty}^{\infty} dx\, P_X(x)\, e^{-ikx} = \left(\int_{-\infty}^{\infty} dx\, P_X(x)\, e^{ikx} \right)^* = f_X(k)^*$$

$$|\, f_X(k)\, | = \left| \int_{-\infty}^{\infty} dx\, P_X(x)\, e^{ikx} \right| \tag{4.15d}$$

$$\le \int_{-\infty}^{\infty} dx\, |P_X(x)\, e^{ikx}| \qquad [\text{ Schwarz inequality used. }]$$

$$= \int_{-\infty}^{\infty} dx\, P_X(x) = 1 \qquad [\, P_X(x) \ge 0 \ \text{ used. }] \tag{4.15e}$$

Taking the inverse Fourier transform of(4.15b), we have

$$P_X(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} f_X(k)\, e^{-ikx} \tag{4.16}$$

Now, differentiating (4.15) gives

$$-i \frac{d f_X(k)}{dk} = \sum_{n=1}^{\infty} \frac{(ik)^{n-1} \langle x^n \rangle}{(n-1)!} = \langle x \rangle + \sum_{n=2}^{\infty} \frac{(ik)^{n-1} \langle x^n \rangle}{(n-1)!} = \langle x \rangle + \sum_{n=1}^{\infty} \frac{(ik)^n \langle x^{n+1} \rangle}{n!}$$

Another differentiation gives

$$(-i)^2 \frac{d^2 f_X(k)}{dk^2} = \sum_{n=1}^{\infty} \frac{(ik)^{n-1} \langle x^{n+1} \rangle}{(n-1)!} = \langle x^2 \rangle + \sum_{n=1}^{\infty} \frac{(ik)^n \langle x^{n+2} \rangle}{n!}$$

which can be generalized to give

$$(-i)^m \frac{d^m f_X(k)}{dk^m} = \langle x^m \rangle + \sum_{n=1}^{\infty} \frac{(ik)^n \langle x^{n+m} \rangle}{n!}$$

Hence,

$$\langle x^m \rangle = (-i)^m \left. \frac{d^m f_X(k)}{dk^m} \right|_{k=0} \tag{4.17}$$

which provides a convenient way to obtain the moments given $f_X(k)$.

Another expansion of $f_X(k)$ that gives a much better rate of convergence is the cumulant expansion

$$f_X(k) = \exp\left[\sum_{n=1}^{\infty} \frac{(ik)^n}{n!} C_n(X) \right] \tag{4.18}$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!} \Big[\sum_{n=1}^{\infty} \frac{(i\,k)^n}{n!} C_n(X) \Big]^m \qquad (4.18a)$$

where $C_n(X)$ is called the nth cumulant.

If we expand (4.18a) and compare the coefficient of k^n with that in (4.15), we obtain an expression of $\langle x^n \rangle$ in terms of a linear combination of $\{ C_k(X) ;\ k = 1,\ \ldots, n \}$.

Alternatively, we can write (4.18) as

$$\sum_{n=1}^{\infty} \frac{(i\,k)^n}{n!} C_n(X) = \ln f_X(k)$$

$$= \ln\Big[\sum_{m=0}^{\infty} \frac{(i\,k)^m \langle x^m \rangle}{m!} \Big] \qquad [\ (4.15) \text{ used.}\ ]$$

$$= \ln\Big[1 + \sum_{m=1}^{\infty} \frac{(i\,k)^m \langle x^m \rangle}{m!} \Big]$$

$$= \sum_{p=1}^{\infty} \frac{(-)^{p+1}}{p} \Big[\sum_{m=1}^{\infty} \frac{(i\,k)^m \langle x^m \rangle}{m!} \Big]^p \qquad (4.18b)$$

If we expand the R.H.S. of (4.18b) and compare the coefficient of k^n with that on the L.H.S., we obtain an expression of $C_n(X)$ in terms of a linear combination of $\{ \langle x^k \rangle ;\ k = 1,\ \ldots, n \}$. The result for the first 5 cumulants are [see §Code]

$$C_1[X] == \langle x \rangle$$
$$C_2[X] == -\langle x \rangle^2 + \langle x^2 \rangle$$
$$C_3[X] == 2\langle x \rangle^3 - 3\langle x \rangle \langle x^2 \rangle + \langle x^3 \rangle \qquad (4.19\text{-}22)$$
$$C_4[X] == -6\langle x \rangle^4 + 12\langle x \rangle^2 \langle x^2 \rangle - 3\langle x^2 \rangle^2 - 4\langle x \rangle \langle x^3 \rangle + \langle x^4 \rangle$$
$$C_5[X] == 24\langle x \rangle^5 - 60\langle x \rangle^3 \langle x^2 \rangle + 30\langle x \rangle \langle x^2 \rangle^2 + 20\langle x \rangle^2 \langle x^3 \rangle - 10\langle x^2 \rangle \langle x^3 \rangle - 5\langle x \rangle \langle x^4 \rangle + \langle x^5 \rangle$$

Code

In[∘]:=
```
Cn[n_, k_] := (n! / i^n Coefficient[Sum[(-1)^(p+1)/p (Sum[(i k)^m x[m] / m!, {m, 1, n}])^p, {p, 1, n}], k, n]) // ExpandAll
```

In[∘]:=
```
Cn[3, k]
```

Out[∘]= $2\,x[1]^3 - 3\,x[1] \times x[2] + x[3]$

In[∘]:=
```
Cn[3, k] /. x[i_] → ⟨x^i⟩
```

Out[∘]= $2\langle x \rangle^3 - 3\langle x \rangle \langle x^2 \rangle + \langle x^3 \rangle$

In[∘]:=
```
Table[C_i[X] == (Cn[i, k] /. x[j_] → ⟨x^j⟩), {i, 5}] // TableForm
```

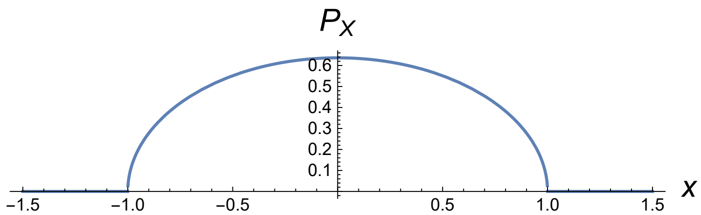
Out[∘]//TableForm=

$$C_1[X] == \langle x \rangle$$
$$C_2[X] == -\langle x \rangle^2 + \langle x^2 \rangle$$
$$C_3[X] == 2\langle x \rangle^3 - 3\langle x \rangle \langle x^2 \rangle + \langle x^3 \rangle$$
$$C_4[X] == -6\langle x \rangle^4 + 12\langle x \rangle^2 \langle x^2 \rangle - 3\langle x^2 \rangle^2 - 4\langle x \rangle \langle x^3 \rangle + \langle x^4 \rangle$$
$$C_5[X] == 24\langle x \rangle^5 - 60\langle x \rangle^3 \langle x^2 \rangle + 30\langle x \rangle \langle x^2 \rangle^2 + 20\langle x \rangle^2 \langle x^3 \rangle - 10\langle x^2 \rangle \langle x^3 \rangle - 5\langle x \rangle \langle x^4 \rangle + \langle x^5 \rangle$$

Ex.4.6.

Consider the probability density for the circular distribution

$$P_X(x) = \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & \text{for } |x| \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$



Find $f_X(k)$ and the 1st four moments and cumulants.

Answer

$$f_X(k) = \frac{2}{\pi}\int_{-1}^{1} dx\, e^{ikx}\sqrt{1-x^2} = \frac{2}{k}J_1(k) \qquad \text{[See §Code.]}$$

where $J_n(x)$ is the n^{th} order Bessel function of the 1st kind. To $O(k^6)$, we have

$$f_X(k) = 1 - \frac{k^2}{8} + \frac{k^4}{192} + O[k]^6$$

The 1st four moments are [see §Code]

$$\langle x \rangle == 0$$
$$\langle x^2 \rangle == \frac{1}{4}$$
$$\langle x^3 \rangle == 0$$
$$\langle x^4 \rangle == \frac{1}{8}$$

The 1st four cumulants are

$$C_1[X] == 0$$
$$C_2[X] == \frac{1}{4}$$
$$C_3[X] == 0$$
$$C_4[X] == -\frac{1}{16}$$

Code

In[1]:=
```
PX[x_] := { 2/π Sqrt[1 - x^2]   -1 ≤ x ≤ 1
          { 0                   True
```

In[7]:=
```
Plot[PX[x], {x, -1.5, 1.5},
    AxesLabel → {"x", "PX"},
    AspectRatio → Automatic
]
```

In[ ]:=
```
PX[x_] := 2/π Sqrt[1 - x^2]
```

In[∘]:=
```
FX[k_] = ∫_{-1}^{1} e^(ⅈ k x) PX[x] ⅆx
```

Out[∘]= $\frac{2\,\text{BesselJ}[1, k]}{k}$

In[∘]:=
```
FX[k] + O[k]^6
```

Out[∘]= $1 - \frac{k^2}{8} + \frac{k^4}{192} + O[k]^6$

In[∘]:=
```
μ[n_] := ∫_{-1}^{1} PX[x] x^n ⅆx // Simplify
```

In[∘]:=
```
Table[⟨x^n⟩ == μ[n], {n, 4}] // TableForm
```

Out[∘]//TableForm=

$\langle x \rangle == 0$
$\langle x^2 \rangle == \frac{1}{4}$
$\langle x^3 \rangle == 0$
$\langle x^4 \rangle == \frac{1}{8}$

In[∘]:=
```
Table[C_i[X] == (Cn[i, k] /. x[j_] → μ[j]), {i, 4}] // TableForm
```

Out[∘]//TableForm=

$C_1[X] == 0$
$C_2[X] == \frac{1}{4}$
$C_3[X] == 0$
$C_4[X] == -\frac{1}{16}$