4.E.2. The Gaussian (or Normal) Distribution

As shown in Fig.4.2, increasing N makes the graph of $P_{Y^{(N)}}(y)$ look more and more like a continuous probability density function if the plot range is kept at $0 \le y \le N$. However, the spacing Δy between realizations stays equal to $\Delta n_1 = 1$ even as $N \to \infty$; thus keeping $P_{Y^{(N)}}(y)$ discrete mathematically.

We can turn $P_{Y^{(N)}}(y)$ into a truly continuous density by using a realization spacing that diminishes as N increases. Since the standard deviation is a measure of the "spread" of the density function, it is the perfect candidate for the purpose. Therefore, we introduce the (central) stochastic variable

$$Z^{(N)} \equiv \frac{Y^{(N)} - \langle y \rangle}{\sigma_{Y^{(N)}}} \qquad (4.49a)$$

$$= \frac{Y^{(N)} - N p}{\sqrt{N p q}} \qquad \text{[(4.45-6) used.]} \qquad (4.49b)$$

$$\to \quad z = \frac{y - N p}{\sqrt{N p q}} = \frac{n_1 - N p}{\sqrt{N p q}} \qquad (4.49c)$$

$$\therefore \quad \Delta z = \frac{\Delta n_1}{\sqrt{N p q}} = \frac{1}{\sqrt{N p q}} \xrightarrow[N \to \infty]{} 0 \qquad \text{[As promised.]}$$

Note: Use of a central variable makes $\langle z \rangle = 0$, which is optional.

Using (4.34) & (4.49c), we have

$$P_{Z^{(N)}}(z) = \int_{-\infty}^{\infty} d y \, P_{Y^{(N)}}(y) \, \delta\left(z - \frac{y - N p}{\sqrt{N p q}} \right) \qquad (4.49)$$

and

$$f_{Z^{(N)}}(k) = \int_{-\infty}^{\infty} d z \, e^{i k z} P_{Z^{(N)}}(z)$$

$$= \int_{-\infty}^{\infty} d z \, e^{i k z} \int_{-\infty}^{\infty} d y \, P_{Y^{(N)}}(y) \, \delta\left(z - \frac{y - N p}{\sqrt{N p q}} \right)$$

$$= \int_{-\infty}^{\infty} d y \, P_{Y^{(N)}}(y) \exp\left(i k \frac{y - N p}{\sqrt{N p q}} \right)$$

$$= \exp\left(-i k \frac{N p}{\sqrt{N p q}} \right) \int_{-\infty}^{\infty} d y \, P_{Y^{(N)}}(y) \exp\left(i \frac{k}{\sqrt{N p q}} y \right)$$

$$= \exp\left(-i k \sqrt{\frac{N p}{q}} \right) f_{Y^{(N)}}\left(\frac{k}{\sqrt{N p q}} \right)$$

$$= e^{-i k \sqrt{N p / q}} \left(q + p \, e^{i k / \sqrt{N p q}} \right)^N \qquad \text{[(4.41) used.]}$$

which, with the help of

$$-\sqrt{\frac{p}{N q}} + \sqrt{\frac{1}{N p q}} = \sqrt{\frac{q}{N p}} \left(-\frac{p}{q} + \frac{1}{q} \right) = \sqrt{\frac{q}{N p}} \qquad [\, p + q = 1 \text{ used. }]$$

becomes

$$f_{Z^{(N)}}(k) = \left(q \, e^{-i k \sqrt{p / N q}} + p \, e^{i k \sqrt{q / N p}} \right)^N \qquad (4.50)$$

Next, we use

$$\left(1+\frac{z}{N}\right)^N = \sum_{n=0}^{\infty} \frac{N!}{n!\,(N-n)!}\left(\frac{z}{N}\right)^n \qquad \text{[Binomial expansion.]}$$
$$= \sum_{n=0}^{\infty} \frac{N!}{N^n (N-n)!}\,\frac{z^n}{n!}$$
$$\xrightarrow[N\to\infty]{} \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \left[\ \frac{N!}{(N-n)!} = \overbrace{N\,(N-1)\ldots(N-n+1)}^{n\text{ terms}} \xrightarrow[N\to\infty]{} N^n\ \right]$$
$$= e^z \tag{4.54}$$

to evaluate (4.50) for $N \to \infty$.

To begin, we expand the exponentials to get

$$f_{Z^{(N)}}(k) = \left\{ q\left[1 - i\,k\sqrt{\frac{p}{N\,q}} - \frac{k^2 p}{2\,N\,q} + \sum_{n=3}^{\infty}\frac{1}{n!}\left(-i\,k\sqrt{\frac{p}{N\,q}}\right)^n\right]\right.$$
$$\left. + p\left[1 + i\,k\sqrt{\frac{q}{N\,p}} - \frac{k^2 q}{2\,N\,p} + \sum_{n=3}^{\infty}\frac{1}{n!}\left(i\,k\sqrt{\frac{q}{N\,p}}\right)^n\right]\right\}^N$$
$$= \left(1 - \frac{k^2}{2\,N} + R_N\right)^N \tag{4.51}$$

where

$$R_N = \sum_{n=3}^{\infty}\frac{(i\,k)^n}{n!\,N^{n/2}}\left[(-)^n\, q\left(\frac{p}{q}\right)^{n/2} + p\left(\frac{q}{p}\right)^{n/2}\right] \tag{4.52}$$

Since R_N is $O(N^{-3/2})$, it makes no contribution when we apply (4.54) to (4.51). Therefore,

$$f_Z(k) \equiv \lim_{N\to\infty} f_{Z^{(N)}}(k) = e^{-k^2/2} \tag{4.53}$$

$$\to \quad P_Z(z) = \int_{-\infty}^{\infty}\frac{d\,k}{2\,\pi}\, e^{-i\,k\,z}\, f_Z(k)$$
$$= \int_{-\infty}^{\infty}\frac{d\,k}{2\,\pi}\, e^{-i\,k\,z}\, e^{-k^2/2}$$
$$= e^{-z^2/2}\int_{-\infty}^{\infty}\frac{d\,k}{2\,\pi}\, e^{-(k+i\,z)^2/2}$$
$$= \frac{1}{\sqrt{2\,\pi}}\, e^{-z^2/2} \tag{4.55}$$

$\therefore \quad \langle z\rangle = 0 \qquad \to \qquad \mathrm{Var}(Z) = \langle z^2\rangle = 1 = \sigma_Z$ [See §Code.]

Using (4.49a), we have

$$P_Y(y) \equiv \lim_{N\to\infty} P_{Y^{(N)}}(y) = \int_{-\infty}^{\infty} d\,z\, P_Z(z)\ \delta\left(y - \langle y\rangle - \sigma_Y\, z\right)$$
$$= \frac{1}{\sigma_Y}\int_{-\infty}^{\infty} d\,z\ \frac{1}{\sqrt{2\,\pi}}\, e^{-z^2/2}\ \delta\left[z - \frac{1}{\sigma_Y}\left(y - \langle y\rangle\right)\right]$$
$$= \frac{1}{\sigma_Y\sqrt{2\,\pi}}\exp\left[-\frac{(y - \langle y\rangle)^2}{2\,\sigma_Y^2}\right] \tag{4.56}$$

which is the Gaussian probability density (or distribution) in terms of non-central moments.

Code

In[•]:=
```
Assuming[ z > 0, 1/(2 π) ∫_{-∞}^{∞} e^(-i k z-k^2/2) dk]
```

Out[•]= $\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}$

```
(*  〈z〉  &  〈z^2〉  *)
1 / √(2 π) ∫_{-∞}^{∞} e^(-z^2/2) # dz & /@ {z, z^2}
```

Out[•]= {0, 1}