4.E.3. The Poisson Distribution

The Poisson distribution is simply the binomial distribution for

$$N \to \infty \qquad \& \qquad p \to 0 \qquad \text{such that} \qquad N p = a \neq 0 \tag{4.57a}$$

Putting (4.57a) into (4.38) gives

$$\begin{aligned} P(n_1) &= \lim_{N\to\infty} \frac{N(N-1)\dots(N-n_1+1)}{n_1!}\left(1-\frac{a}{N}\right)^{N-n_1}\left(\frac{a}{N}\right)^{n_1} \\ &= \lim_{N\to\infty} \frac{a^{n_1}}{n_1!}\left(1-\frac{a}{N}\right)^{N} \qquad [\; N-m \to N \quad \forall\; m \text{ finite} \;] \\ &= \frac{a^{n_1}}{n_1!} e^{-a} \qquad [\text{ 4.54) used. }] \end{aligned} \tag{4.60}$$

which is known as the Poisson distribution.

The normalization condition is

$$\sum_{n_1=0}^{\infty} P(n_1) = e^{-a} \sum_{n_1=0}^{\infty} \frac{a^{n_1}}{n_1!} = e^{-a} e^{a} = 1 \tag{4.60a}$$

as expected.

The 1st moment is

$$\langle n_1 \rangle = e^{-a} \sum_{n_1=0}^{\infty} n_1 \frac{a^{n_1}}{n_1!} = e^{-a} \sum_{n_1=1}^{\infty} \frac{a^{n_1}}{(n_1-1)!} = a\, e^{-a} \sum_{n_1=0}^{\infty} \frac{a^{n_1}}{n_1!} = a \tag{4.60b}$$

in agreement with (4.57a).

Putting (4.57a) into (4.41) gives

$$\begin{aligned} f_Y(k) &\equiv \lim_{N\to\infty} f_{Y(N)}(k) \\ &= \lim_{N\to\infty} \left[1 + \frac{a}{N}\left(e^{ik}-1\right)\right]^N \\ &= \exp\left[a\left(e^{ik}-1\right)\right] \qquad [\text{ 4.54) used. }] \\ &= e^{-a} \sum_{n=0}^{\infty} \frac{a^n}{n!} e^{ink} \end{aligned} \tag{4.58}$$

Thus

$$\begin{aligned} P(y) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-iky} f_Y(k) \\ &= e^{-a} \sum_{n=0}^{\infty} \frac{a^n}{n!} \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-iky} e^{ink} \qquad [\text{ (4.58) used. }] \\ &= e^{-a} \sum_{n=0}^{\infty} \frac{a^n}{n!} \delta(y-n) \end{aligned} \tag{4.59}$$

which corresponds to a discrete probability assignment [c.f. (4.8)]

$$P_n = e^{-a} \frac{a^n}{n!} \qquad \text{for} \quad n = 0, 1, \dots, \infty \tag{4.59a}$$

that is just the Poisson distribution.

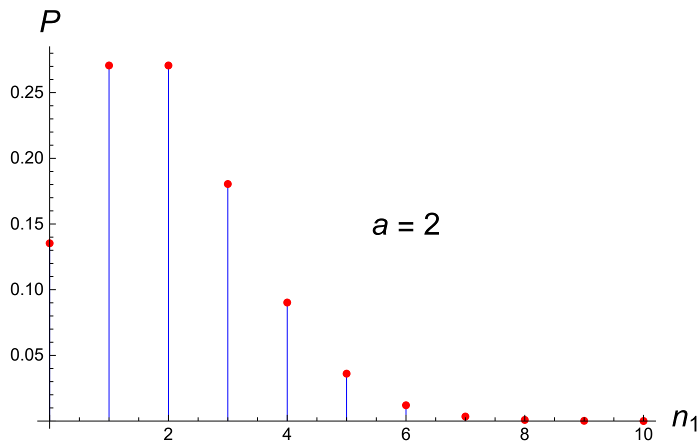

Fig.4.3. Poisson distribution for $a = 2$.

Code

```
P[n_, a_] := e^-a a^n/n!
```

```
In[•]:= (* Fig.4.3. *)
n = 10; a = 2;
ListPlot[ Table[{n1, P[n1, a]}, {n1, 0, n}],
     AxesLabel → {"n1", "P" }, PlotStyle → Red,
     Filling → Bottom, FillingStyle → {Thick, Blue},
     Prolog → Text["a = 2", {6, .15}]
]
```

Ex.4.11.

In the bombardment by a neutron beam upon a thin (one atom thick) gold foil, we can consider the passing through the foil by a neutron to be a single trial. In which case, the chance p of an actual collision between the neutron & an atom is very small. The passing through of N neutrons per unit area per unit time can be considered an event of N trials. Since N is a huge number, the Poisson distribution is applicable.

Assume the average number of collisions for each event to be 2.

(a) What is the probability that no collision occurs in one event?

(b) What is the probability that two collisions occur in one event?

Answer

Before we proceed, it seems appropriate to give an order of magnitude estimate of the situation.

Since gold has an atomic weight of about 200 & mass density of about $20\ \text{gm}/\text{cm}^3$, its molar volume is roughly $10\ \text{cm}^3$. The number of atoms per surface area is therefore

$$\rho_s \approx \left(6\times10^{23} / 10\ \text{cm}^3\right)^{2/3} \approx 7\times10^{15}\ \text{cm}^{-2}$$

The neutron cross section is about $\sigma \approx 8\,\text{barns} = 8\times 10^{-24}\,\text{cm}^2$. The probability for a collision as a neutron passes through the gold foil is therefore

$$p \approx \rho_s\,\sigma \approx 6\times 10^{-8}$$

Since the flux J of the neutron beam is numerically equal to N, the average number of collisions in each event is therefore

$$\langle n\rangle = \rho_s\,\sigma\,J \approx 6\times 10^{-8}\,N$$

Setting $\langle n\rangle = 2$ gives $N \approx 3\times 10^7$ and $J = 3\times 10^7\,\text{cm}^{-2}\cdot s^{-1}$.

The Poisson distribution for the situation is

$$P(n) = \frac{2^n}{n!}\,e^{-2}$$

where n is the number of collisions in one event.

(a) $P(0) = e^{-2} \approx 0.135$

(b) $P(2) = 2\,e^{-2} \approx 0.271$

```
In[ ]:= e^-2.
Out[ ]= 0.135335

In[ ]:= 2 e^-2.
Out[ ]= 0.270671
```