4.E.4. Binomial Random Walk

Consider the 1-D random walk problem. During each time interval τ, a walker must take a step of fixed length Δ either to the left with probability $q = \frac{1}{2}$, or to the right with probability $p = \frac{1}{2}$. The problem is to find the probability distribution for the net displacement of the walker after N steps (or at time $N\tau$).

Let the displacement of the i^{th} step be represented by the stochastic variable X_i with realizations $\pm\Delta$. The corresponding probability density is

$$P_{X_i}(x_i) = p\,\delta(x_i - \Delta) + q\,\delta(x_i + \Delta) = \frac{1}{2}\delta(x_i - \Delta) + \frac{1}{2}\delta(x_i + \Delta)$$

(4.61a)

which gives the characteristic function

$$\begin{aligned} f_{X_i}(k) &= \frac{1}{2}\int_{-\infty}^{\infty} d\,x_i\, e^{i\,k\,x}\left[\,\delta(x_i - \Delta) + \delta(x_i + \Delta)\,\right] \\ &= \frac{1}{2}\left(e^{i\,k\,\Delta} + e^{-i\,k\,\Delta}\right) \\ &= \cos k\,\Delta \end{aligned}$$

(4.61b)

For the event of N consecutive steps, the net displacement is represented by the stochastic variable

$$Y^{(N)} = X_1 + \ldots + X_N$$

Since $\{X_j\}$ are statistically independent,

$$P_{Y^{(N)}}(y) = P_{X_1}(x_1)\ldots P_{X_N}(x_N)$$

and the characteristic function is simply [c.f. (4.41)]

$$\begin{aligned} f_{Y^{(N)}}(k) &= \int_{-\infty}^{\infty} d\,x_1 \ldots \int_{-\infty}^{\infty} d\,x_N\, e^{i\,k\,(x_1 + \ldots + x_N)}\, P_{X_1}(x_1)\ldots P_{X_N}(x_N) \\ &= f_{X_1}(k)\ldots f_{X_N}(k) \\ &= \left(\cos k\,\Delta\right)^N \end{aligned}$$

(4.61c)

$$\begin{aligned} &= \left(1 - \frac{1}{2}k^2\,\Delta^2 + \ldots\right)^N \\ &= 1 - \frac{1}{2}N\,k^2\,\Delta^2 + \ldots \end{aligned}$$

(4.61)

Using (4.17) to calculate the moments, we get

$$\langle y\rangle = -i\,\frac{\partial f_{Y^{(N)}}(k)}{\partial k}\bigg|_{k=0} = i\,N\,\Delta\left(\sin k\,\Delta\right)\left(\cos k\,\Delta\right)^{N-1}\bigg|_{k=0} = 0$$

$$\begin{aligned} \langle y^2\rangle &= -i\,\frac{\partial}{\partial k}\left[\,i\,N\,\Delta\left(\sin k\,\Delta\right)\left(\cos k\,\Delta\right)^{N-1}\right]\bigg|_{k=0} \\ &= N\,\Delta\left[\,\Delta\left(\cos k\,\Delta\right)\left(\cos k\,\Delta\right)^{N-1} + (N-1)\,\Delta\left(\sin k\,\Delta\right)^2\left(\cos k\,\Delta\right)^{N-2}\right]\bigg|_{k=0} \\ &= N\,\Delta^2 \end{aligned}$$

$\rightarrow$ $\mathrm{Var}(y) = N\,\Delta^2$ & $\sigma_{Y^{(N)}} = \sqrt{N}\,\Delta$

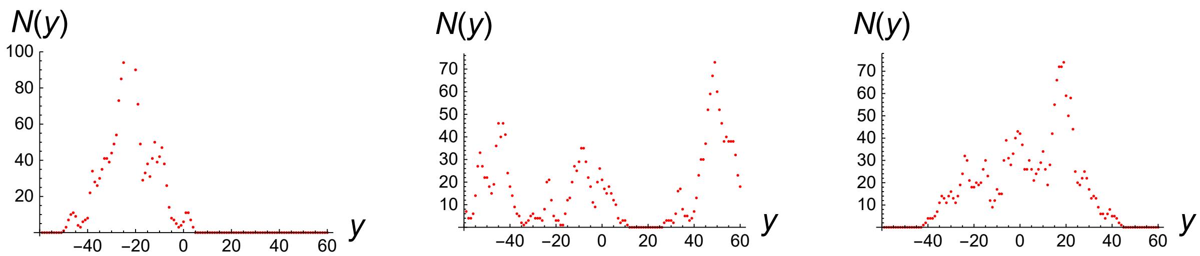

Fig.4.4. Results of three realizations of the random walk discussed above.
$N(y)$ is the number of times the walker reaches point y.
The walker is assumed to start at $y = 0$ and

$$N = 2000 \qquad \Delta = 1 \qquad p = q = \frac{1}{2}$$

Extension of the random walk to higher spatial dimensions is straightforward. After which, it can be used as a model for phenomena such as Brownian motion and diffusion. For such purposes, it would be helpful to obtain the (differential) equation of motion of the problem.

Setting $t = N\tau$, we can treat N as a time variable so that

$$Y^{(N)} = Y(t) \qquad P_{Y^{(N)}}(y) = P_Y(y, t) \qquad f_Y(k) = f_Y(k, t) \tag{4.62a}$$

In the continuum limit, i.e.,

$$N \to \infty \qquad \tau \to 0 \qquad \text{such that} \qquad t = N\tau \neq 0$$

(4.62b)

$f_Y(k, t)$ becomes a continuous function of t so that we can derive a differential equation satisfied by it.

Thus,

$$\frac{\partial f_Y(k, t)}{\partial t} = \lim_{N\to\infty,\, \tau\to 0} \frac{f_{Y^{(N)}}\left[k, (N+1)\tau\right] - f_{Y^{(N)}}(k, N\tau)}{\tau}$$

(4.63)

$$= \lim_{N\to\infty,\, \tau\to 0} \frac{\left(\cos k\Delta - 1\right) f_{Y^{(N)}}(k, N\tau)}{\tau} \qquad [\,(4.61c) \text{ used.}\,]$$

$$= \lim_{N\to\infty,\, \tau\to 0,\, \Delta\to 0} \frac{-\left(k\Delta\right)^2 f_{Y^{(N)}}(k, N\tau)}{2\tau}$$

(4.64a)

where we have applied the spatial continuum limit to set $\Delta \to 0$. In order to keep (4.64a) finite, we set

$$D = \lim_{\tau\to 0,\, \Delta\to 0} \frac{\Delta^2}{2\tau}$$

(4.64b)

so that (4.64a) becomes

$$\frac{\partial f_Y(k, t)}{\partial t} = -D k^2 f_Y(k, t)$$

(4.65)

Taking the inverse Fourier transform, we get the equation of motion we searched for:

$$\frac{\partial P_Y(y, t)}{\partial t} = D \frac{d^2 P_Y(y, t)}{d y^2}$$

(4.65a)

If we interpret $P_Y(y, t)$ as the particle density, (4.65a) is just the 1-D diffusion equation, with D being the diffusion coefficient.

For the initial condition

$$f_Y(k, 0) = 1$$

the solution to (4.65) is easily verified to be

$$f_Y(k, t) = e^{-D k^2 t}$$

(4.66)

Taking the inverse Fourier transform, we get

$$P_Y(y, t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-iky} e^{-D k^2 t}$$
$$= \frac{1}{\sqrt{4\pi D t}} \exp\left(-\frac{y^2}{4 D t}\right) \qquad \text{[See §Code]}$$

(4.67)

which is a Gaussian with $\sigma_Y = \sqrt{2 D t}$. Thus, $P_Y(y, t)$ spreads out with time, as befits a diffusion.

(4.67) can be verified as a solution to (4.65a) [see §Code].

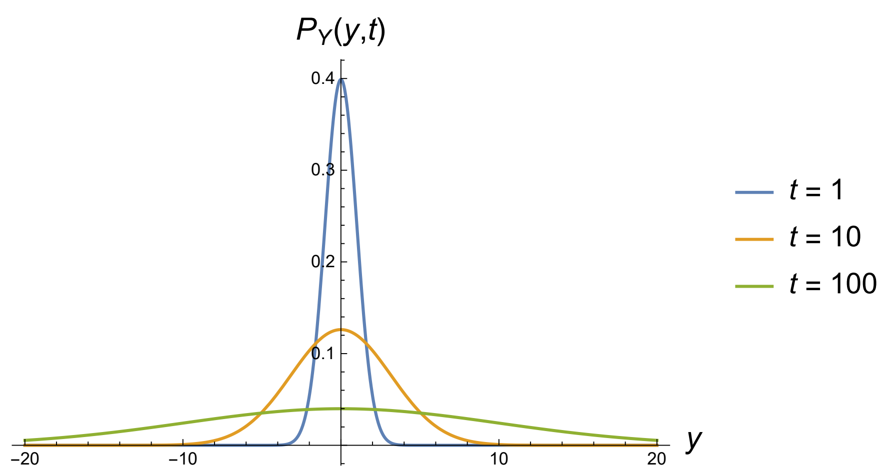

Fig.4.5. Spreading of $P_Y(y, t)$ with time for $D = \frac{1}{2}$.

To emphasize the spatial continuum limit, we introduce the stochastic variable

$$Z^{(N)} = \frac{Y^{(N)}}{\Delta \sqrt{N}} \qquad \& \qquad z = \frac{y}{\Delta \sqrt{N}} = \frac{y}{\Delta \sqrt{t/\tau}} \xrightarrow[\Delta, \tau \to 0]{} \frac{y}{\sqrt{2 D t}} \qquad \text{[(4.64b) used.]}$$

(4.68a)

Using

$$P_{Z^{(N)}}(z) = \int_{-\infty}^{\infty} dy\, \delta\left(z - \frac{y}{\Delta \sqrt{N}}\right) P_{Y^{(N)}}(y)$$

(4.68b)

we have

$$f_{Z^{(N)}}(k) = \int_{-\infty}^{\infty} dz\, e^{ikz} P_{Z^{(N)}}(z)$$
$$= \int_{-\infty}^{\infty} dz\, e^{ikz} \int_{-\infty}^{\infty} dy\, \delta\left(z - \frac{y}{\Delta\sqrt{N}}\right) P_{Y^{(N)}}(y)$$
$$= \int_{-\infty}^{\infty} dy\, e^{iky/\Delta\sqrt{N}} P_{Y^{(N)}}(y)$$
$$= f_{Y^{(N)}}\left(\frac{k}{\Delta\sqrt{N}}\right)$$
$$= \left[\cos\left(\frac{k}{\sqrt{N}}\right)\right]^N \qquad \text{[(4.61c) used.]}$$

(4.68)

which is the same as (4.50) with $p = q = \frac{1}{2}$. Therefore, we can use (4.53) to write

$$f_Z(k) = \lim_{N\to\infty} f_{Z^{(N)}}(k) = e^{-k^2/2}$$

(4.69)

so that [see (4.55)]

$$P_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

(4.70)

Hence,

$$P_{Y^{(N)}}(y) = \int_{-\infty}^{\infty} dz\, P_{Z^{(N)}}(z)\, \delta\left(y - \Delta\sqrt{N}\, z\right)$$
$$\approx \int_{-\infty}^{\infty} dz\, P_Z(z)\, \delta\left(y - \Delta\sqrt{N}\, z\right) \qquad \text{for } N >> 1$$
$$= \frac{1}{\Delta\sqrt{N}} \int_{-\infty}^{\infty} dz\, P_Z(z)\, \delta\left(z - \frac{y}{\Delta\sqrt{N}}\right)$$
$$= \frac{1}{\Delta\sqrt{2\pi N}} \exp\left(-\frac{y^2}{2\Delta^2 N}\right) \qquad \text{[(4.70) used.]}$$

(4.70)

$$\therefore \quad P_Y(y, t) = P_{Y^{(N)}}(y)\Big|_{N\Delta^2 = 2Dt} \qquad \text{[(4.62b) \& (4.64b) combined.]}$$
$$= \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{y^2}{4Dt}\right) \qquad (4.71)$$

in agreement with (4.67).

Code: Fig.4.4.

```
In[∘]:= ranWalk[Ns_, maxi_] := Module[ {Ny, y},
        Ny = Table[0, {i, -maxi, maxi}];
        y = 0;
        Do[
             y = y + 2 RandomInteger[] - 1;
             Ny[[y + maxi]] = Ny[[y + maxi]] + 1,
             {n, Ns}
         ];
        Table[{i, Ny[[i + maxi]]}, {i, -maxi, maxi}]
   ]
```

Caution: Errors occur if walker wanders beyond $|y| = \text{maxi}$.

```
In[∘]:= Ns = 2000; maxi = 60;
        Ny = ranWalk[Ns, maxi];

In[∘]:= ListPlot[Ny, AxesOrigin → {-maxi, 0},
           PlotStyle → Red,
           AxesLabel → {"y", "N(y)"}]

In[∘]:= Ny = ranWalk[Ns, maxi];
        ListPlot[Ny, AxesOrigin → {-maxi, 0},
           PlotStyle → Red,
           AxesLabel → {"y", "N(y)"}]

In[∘]:= Ny = ranWalk[Ns, maxi];
        ListPlot[Ny, AxesOrigin → {-maxi, 0},
           PlotStyle → Red,
           AxesLabel → {"y", "N(y)"}]
```

Code

```
(* f_Y(y) *)
fY = e^(-d k^2 t)
```

Out[∘]= $e^{-d k^2 t}$

```
(* P_Y(y) *)
PY = Assuming[ d > 0 && t > 0, 1/(2 π) ∫_{-∞}^{∞} e^(-i k y) fY dk]
```

Out[∘]= $$\frac{e^{-\frac{y^2}{4 d t}}}{2\sqrt{\pi}\,\sqrt{d\,t}}$$

```
(* Shows PY is solution to diffusion eq. *)
∂_t PY == d ∂_{y,y} PY // Simplify
```

Out[∘]= True

In[ ]:=
```
(* Fig.4.5 *)
ls = PY /. d → 1/2 /. t → {1, 10, 100};
Plot[ls, {y, -20, 20},
    PlotRange → All,
    AxesLabel → {"y", "PY (y,t) "},
    PlotLegends → { "t = 1", "t = 10", "t = 100"}
]
```