4.F. The Central Limit Theorem and The Law of Large Numbers

The central limit theorem states that:

The distribution of the outcomes of events always approaches the Gaussian form as the number of events increases indefinitely, provided the moments of the individual events are finite.

A full proof of the theorem is given in §S4.C. Here, we present a weaker proof that is applicable to measurements, which are based on the law of large numbers.

4.F.1. The Central Limit Theorem

In the following "error analysis", we shall treat all experimental measurements as random experiments. However, the quantity to be measured is assumed to be deterministic, i.e., it gives the same value under identical experimental settings. However, due to (small) uncertainties in the implementation of these "identical" settings, measured values exhibit fluctuations that can, in principle, be reduced by improving the accuracy of the instruments.

Let X be the stochastic variable that is being measured. After N "identical" measurements, the measured value $x_M^{(N)}$ of X is represented as

$$x_M^{(N)} = \langle x \rangle^{(N)} \pm \sigma_X^{(N)} \tag{4.72a}$$

Since X is actually deterministic, we expect the quantities in (4.72a) to converge to some fixed values. In fact, the law of large numbers gives

$$x_M = \langle x \rangle \qquad \text{for} \quad N \to \infty \qquad \left[\lim_{N \to \infty} \sigma_X^{(N)} = 0 \right] \tag{4.72b}$$

The error in the j^{th} measurement is given by

$$Z_j = X_j - \langle x \rangle \qquad\qquad z_j = x_j - \langle x \rangle \tag{4.72c}$$

and, after N measurements, by

$$Y^{(N)} = \frac{1}{N}(X_1 + \ldots + X_N) - \langle x \rangle \tag{4.72d}$$

$$= X^{(N)} - \langle x \rangle \qquad\qquad x^{(N)} = \frac{1}{N}(x_1 + \ldots + x_N) \tag{4.72e}$$

$$= \frac{1}{N}(Z_1 + \ldots + Z_N) \qquad\qquad y^{(N)} = \frac{1}{N}(z_1 + \ldots + z_N) \tag{4.72f}$$

where $X^{(N)}$ is the called the sample average.

Since the measurements are "identical" ,

$$P_{X_j}(x_j) = P_X(z_j) \qquad\qquad P_{Z_j}(z_j) = P_Z(z_j) \qquad\qquad \forall\ j$$

$$\to \quad f_{Z_j}(k) = f_Z(k) = \int_{-\infty}^{\infty} dz\, e^{ikz} P_Z(z) \tag{4.73a}$$

Since the measurements are statistically independent [see (4.41)]

$$f_{Y^{(N)}}(k) = \int_{-\infty}^{\infty} dz_1 \ldots \int_{-\infty}^{\infty} dz_N\, e^{ik(z_1 + \ldots + z_N)/N} P_Z(z_1) \ \ldots\ P_Z(z_N) \qquad \text{[(4.72f) used.]}$$

$$= \left[f_Z\!\left(\frac{k}{N}\right) \right]^N \qquad \text{[(4.73a) used.]} \tag{4.74a}$$

Now,

$$f_Z\left(\frac{k}{N}\right) = \int_{-\infty}^{\infty} dz \left(1 + \frac{i\,k}{N} z - \frac{k^2}{2\,N^2} z^2 + \ldots\right) P_Z(z)$$

$$= 1 - \frac{k^2}{2\,N^2} \langle z^2 \rangle + \ldots \qquad [\ \langle z \rangle = 0 \text{ used (see (4.72c). }]$$

$$= 1 - \frac{k^2}{2\,N^2} \sigma_Z^2 + \ldots \qquad (4.73b)$$

so that (4.74a) becomes

$$f_{Y^{(N)}}(k) = \left(1 - \frac{k^2}{2\,N^2} \sigma_Z^2 + \ldots\right)^N$$

$$\xrightarrow[N\to\infty]{} \exp\left(-\frac{k^2}{2\,N} \sigma_Z^2\right) \qquad [\ (4.54) \text{ used. }] \qquad (4.74)$$

where we have assumed all moments of Z are finite so that $O(N^{-2})$ terms can be dropped.

Taking the inverse Fourier transform gives

$$P_Y(y) = \lim_{N\to\infty} P_{Y^{(N)}}(y) \qquad [\ y = y^{(N)}\ ]$$

$$= \lim_{N\to\infty} \int_{-\infty}^{\infty} \frac{d\,k}{2\,\pi} e^{-i\,k\,y} \exp\left(-\frac{k^2}{2\,N} \sigma_Z^2\right) \qquad [\ (4.74) \text{ used. }]$$

$$= \lim_{N\to\infty} \sqrt{\frac{N}{2\,\pi\,\sigma_Z^2}} \exp\left(-\frac{N\,y^2}{2\,\sigma_Z^2}\right) \qquad (4.75)$$

Since Z is simply the central version of X [see (4.72c)], we have

$$\sigma_Z = \sigma_X \qquad (4.75a)$$

Thus, as $N \to \infty$, the distribution of the error always approaches a Gaussian with a standard deviation

$$\sigma_{Y^{(N)}} = \frac{1}{\sqrt{N}} \sigma_Z = \frac{1}{\sqrt{N}} \sigma_X \qquad (4.75b)$$

where σ_Z is the a standard deviation of error in one measurement. Compared with (4.72a), we have

$$\sigma_X^{(N)} = \sigma_{Y^{(N)}}$$

Code

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In[∘]:= Assuming[y > 0 && n > 0 && σ > 0, 1/(2 π) ∫_{-∞}^{∞} Exp[-ⅈ k y - k^2/(2 n) σ^2] ⅆk]
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$$\text{Out}[\circ]= \frac{e^{-\frac{n\,y^2}{2\,\sigma^2}} \sqrt{n}}{\sqrt{2\,\pi}\ \sigma}$$

4.F.2. The Law of Large Numbers

The weak form of the law of large numbers (or Khinchin’s law) states that

> The sample average converges, in the probabilistic sense, to the mean,
> as the number of measurements increases indefinitely.

i.e.,

$$\lim_{N\to\infty} \text{Prob}\left(\left| x^{(N)} - \langle x\rangle \right| \geq \epsilon \right) = 0 \qquad \forall\ \epsilon > 0 \tag{4.76a}$$

or, using (4.72d),

$$\lim_{N\to\infty} \text{Prob}\left(\left| y^{(N)} \right| \geq \epsilon \right) = 0 \qquad \forall\ \epsilon > 0 \tag{4.76b}$$

Proof of the law is as follows.

Consider the variance for one measurement

$$\text{Var}(X) = \int_{-\infty}^{\infty} d\,x \left(x - \langle x\rangle \right)^2 P_X(x) \tag{4.76}$$

If we take the region $|x - \langle x\rangle| \leq \epsilon$ out from the integral, we get

$$\text{Var}(X) \geq \int_{-\infty}^{\langle x\rangle - \epsilon} d\,x \left(x - \langle x\rangle \right)^2 P_X(x) + \int_{\langle x\rangle + \epsilon}^{\infty} d\,x \left(x - \langle x\rangle \right)^2 P_X(x) \tag{4.77}$$

$$\geq \epsilon^2 \left[\int_{-\infty}^{\langle x\rangle - \epsilon} d\,x\, P_X(x) + \int_{\langle x\rangle + \epsilon}^{\infty} d\,x\, P_X(x) \right] \qquad [\ (x - \langle x\rangle)^2 \geq \epsilon^2 \text{ inside each integral.}\ ]$$

$$\geq \epsilon^2\, \text{Prob}\left(\left| x - \langle x\rangle \right| \geq \epsilon \right) \tag{4.78}$$

$$\to \quad \text{Prob}\left(\left| x - \langle x\rangle \right| \geq \epsilon \right) \leq \left(\frac{\sigma_X}{\epsilon} \right)^2 \tag{4.79}$$

which is known as the Tchebycheff (or Chebyshev) inequality.

After $N >> 1$ measurements, the central theorem (4.75b) gives

$$\text{Prob}\left(\left| y^{(N)} \right| \geq \epsilon \right) \leq \frac{1}{N} \left(\frac{\sigma_X}{\epsilon} \right)^2 \tag{4.80}$$

$$\xrightarrow[N\to\infty]{} 0 \qquad \text{if } \sigma_X \text{ is finite.} \tag{4.81}$$

QED.