

## 5.B. General Theory

The following notations will be used for the various probability densities for the stochastic variable  $Y$ :

$$P_1(y_1, t_1) \equiv P_1(1) \equiv P(1) \equiv \text{probability density for } Y = y_1 \text{ at } t = t_1. \quad (5.1)$$

$$P_2(y_1, t_1; y_2, t_2) \equiv P_2(1, 2) \equiv \text{joint probability density for } Y = y_1 \text{ at } t = t_1 \text{ and } Y = y_2 \text{ at } t = t_2. \quad (5.2)$$

$$P_n(y_1, t_1; \dots; y_n, t_n) \equiv P_n(1, \dots, n) \equiv \text{joint probability density for } Y = y_1 \text{ at } t = t_1, \dots, \text{ and } Y = y_n \text{ at } t = t_n. \quad (5.3)$$

Obviously, every  $P_n$  is symmetric in its arguments:

$$P_n(1, 2, \dots, n) = P_n(2, 1, \dots, n) = \dots \quad (5.3a)$$

Note that all probabilities are positive:

$$P_n \geq 0 \quad (5.4)$$

Since the sum of all possible probabilities of an event is 1, we have

$$\int d y_1 P_1(1) = 1 \quad \forall t_1 \quad (5.6)$$

$$\int d y_n P_n(1, \dots, n) = P_{n-1}(1, \dots, n-1) \quad \forall t_n \quad (5.5)$$

The  $n^{\text{th}}$  **moment** ( or **correlation function** ) of  $Y$  is defined as

$$\langle y(t_1) y(t_2) \dots y(t_n) \rangle \equiv \int d y_1 \dots \int d y_n P_n(1, \dots, n) y_1 y_2 \dots y_n \quad (5.7)$$

A process is called **stationary** if all averages (or moments) are independent of the time origin so that it behaves the same way no matter when we start the measurements. This means every  $P_n$  is invariant if its time arguments are all displaced by the same amount:

$$P_n(y_1, t_1; \dots; y_n, t_n) = P_n(y_1, t_1 + \tau; \dots; y_n, t_n + \tau) \quad \forall n, \tau \quad (5.8)$$

Setting  $\tau = -t_1$ , we have

$$P_1(y_1, t_1) = P_1(y_1, 0) \equiv P_1(y_1) \quad (5.9)$$

so that  $\langle y(t_1) \rangle$  is  $t$ -independent.

Similarly,

$$\begin{aligned} P_2(y_1, t_1; y_2, t_2) &= P_2(y_1, 0; y_2, t_2 - t_1) & (\tau = -t_1) \\ &= P_2(y_1, t_1 - t_2; y_2, 0) & (\tau = -t_2) \end{aligned}$$

so that

$$\begin{aligned} \langle y(t_1) y(t_2) \rangle &= \langle y_1(0) y_2(t_2 - t_1) \rangle \\ &= \langle y_1(t_1 - t_2) y_2(0) \rangle \end{aligned}$$

depends on  $|t_1 - t_2|$ .

By definition, all processes in equilibrium are stationary.

Various  $P_n$ 's can be related by conditional probabilities.

For example,

$$P_2(1, 2) = P_1(1) P_{1|1}(1 | 2) \quad (5.11)$$

where

$$P_{1|1}(1 | 2) \equiv P_{1|1}(y_1, t_1 | y_2, t_2)$$

is the **conditional probability density** that  $Y = y_2$  at  $t = t_2$  given that  $Y = y_1$  at  $t = t_1$ .

Combining (5.5) & (5.11), we have

$$P_1(2) = \int d y_1 P_2(1, 2)$$

$$= \int d y_2 P_1(1) P_{1|1}(1 | 2) \quad (5.12)$$

which relates  $P_1$  of different arguments.

From (5.11), we have

$$\begin{aligned} \int d y_2 P_{1|1}(1 | 2) &= \frac{1}{P_1(1)} \int d y_2 P_2(1, 2) \\ &= \frac{1}{P_1(1)} P_1(1) \\ &= 1 \quad \forall t_1 \end{aligned} \quad (5.13)$$

**Joint conditional probability density** of higher order can also be defined:

$$\begin{aligned} &P_{k|j}(1, \dots, k | k+1, \dots, k+j) \\ &\equiv \text{Joint conditional probability density that } Y = y_{k+1} \text{ at } t = t_{k+1}, \\ &\quad \dots, \text{ and } Y = y_{k+j} \text{ at } t = t_{k+j}, \text{ given that } Y = y_1 \text{ at } t = t_1, \dots, \text{ and} \\ &\quad Y = y_k \text{ at } t = t_k. \end{aligned} \quad (5.14)$$

(5.11) thus generalizes to

$$\begin{aligned} &P_{k+j}(1, \dots, k, k+1, \dots, k+j) \\ &= P_k(1, \dots, k) P_{k|j}(1, \dots, k | k+1, \dots, k+j) \end{aligned} \quad (5.15)$$

Note that  $P_{k|j}(1, \dots, k | k+1, \dots, k+j)$  is invariant under any permutations of the  $k$  ( or  $j$  ) arguments among themselves.

Joint conditional probability densities describe **memory effects**. In this respect, it is convenient to assume that  $t_1 < t_2 < \dots < t_n$  for all  $n$ .

For example, if there are **no memory effects**, then

$$P_{k|1}(1, \dots, k | k+1) = P_1(k+1) \quad \forall k \quad (5.15a)$$

so that

$$\begin{aligned} &P_{k+1}(1, \dots, k, k+1) \\ &= P_k(1, \dots, k) P_1(k+1) \\ &= P_{k-1}(1, \dots, k-1) P_1(k) P_1(k+1) \\ &\quad \vdots \\ &= P_1(1) \dots P_1(k) P_1(k+1) \end{aligned} \quad (5.15b)$$

More generally,

$$P_{k|j}(1, \dots, k | k+1, \dots, k+j) = P_j(k+1, \dots, k+j) \quad \forall k, j$$

so that

$$\begin{aligned} &P_{k+j}(1, \dots, k, k+1, \dots, k+j) \\ &= P_k(1, \dots, k) P_j(k+1, \dots, k+j) \\ &\quad \vdots \\ &= P_1(1) \dots P_1(k) P_1(k+1) \dots P_1(k+j) \end{aligned}$$

A **Markov process** retains memory only of its immediate pass, then

$$P_{k|1}(1, \dots, k | k+1) = P_{1|1}(k | k+1) \quad \forall k \quad (5.16)$$

so that

$$\begin{aligned} &P_{k+1}(1, \dots, k, k+1) \\ &= P_k(1, \dots, k) P_{1|1}(k | k+1) \\ &= P_{k-1}(1, \dots, k-1) P_{1|1}(k-1 | k) P_{1|1}(k | k+1) \\ &\quad \vdots \end{aligned}$$

$$= P_1(1) P_{1|1}(1 | 2) \dots P_{1|1}(k-1 | k) P_{1|1}(k | k+1) \quad (5.17a)$$

Comparing with (5.15b),  $P_{1|1}(k | k+1)$  is called the **transition probability**.

Consider now the special case

$$P_3(1, 2, 3) = P_1(1) P_{1|1}(1 | 2) P_{1|1}(2 | 3) \quad (5.17)$$

$$\rightarrow P_2(1, 3) = \int dy_2 P_3(1, 2, 3) \quad \forall t_2 \quad [ (5.5) \text{ used } ]$$

$$= P_1(1) \int dy_2 P_{1|1}(1 | 2) P_{1|1}(2 | 3) \quad (5.18)$$

On the other hand

$$P_2(1, 3) = P_1(1) P_{1|1}(1 | 3)$$

Comparing with (5.18) then gives

$$P_2(1, 3) = \int dy_2 P_{1|1}(1 | 2) P_{1|1}(2 | 3) \quad (5.19)$$

which is known as the **Chapman-Kolmogorov equation**.