## 5.C.1. Spectral Properties

For a time independent $\boldsymbol{Q}$, we have

$$Q_{mn}(s) = P_{1|1}(m, s \mid n, s+1) = P_{1|1}(m, 0 \mid n, 1) = Q_{mn} \qquad \forall\ s \tag{5.23}$$

or

$$\boldsymbol{Q}(s) = \boldsymbol{Q} \qquad \forall\ s \tag{5.23a}$$

(5.21) can be solved by iteration starting with the self-evident "initial condition"

$$P_{1|1}(m, s \mid n, s) = \delta_{mn} \text{ or } \qquad \boldsymbol{Q}(s \mid s) = \boldsymbol{I} \tag{5.23b}$$

For the 1st time step (or iteration), (5.21) becomes

$$\begin{aligned} P_{1|1}(m, s \mid n, s+1) &= \sum_k P_{1|1}(m, s \mid k, s)\, P_{1|1}(k, s \mid n, s+1) \\ &= \sum_k \delta_{mk}\, P_{1|1}(k, s \mid n, s+1) \\ &= P_{1|1}(m, s \mid n, s+1) \\ &= Q_{mn} \end{aligned}$$

or

$$\boldsymbol{Q}(s \mid s+1) = \boldsymbol{I} \cdot \boldsymbol{Q}(s \mid s+1) = \boldsymbol{Q}$$

For the 2nd time step,

$$\begin{aligned} P_{1|1}(m, s \mid n, s+2) &= \sum_k P_{1|1}(m, s \mid k, s+1)\, P_{1|1}(k, s+1 \mid n, s+2) \\ &= \sum_k Q_{mk}\, Q_{kn} \\ &= (\boldsymbol{Q}^2)_{mn} \end{aligned}$$

or

$$\boldsymbol{Q}(s \mid s+2) = \boldsymbol{Q}(s \mid s+1) \cdot \boldsymbol{Q}(s+1 \mid s+2) = \boldsymbol{Q}^2$$

For the 3rd time step,

$$\begin{aligned} P_{1|1}(m, s \mid n, s+3) &= \sum_k P_{1|1}(m, s \mid k, s+2)\, P_{1|1}(k, s+2 \mid n, s+3) \\ &= \sum_k (\boldsymbol{Q}^2)_{mk}\, Q_{kn} \\ &= (\boldsymbol{Q}^3)_{mn} \end{aligned}$$

or

$$\boldsymbol{Q}(s \mid s+3) = \boldsymbol{Q}(s \mid s+2) \cdot \boldsymbol{Q}(s+2 \mid s+3) = \boldsymbol{Q}^3$$

In general,

$$P_{1|1}\left(m, s_0 \mid n, s\right) = (\boldsymbol{Q}^{s-s_0})_{mn} \tag{5.24}$$

or

$$\boldsymbol{Q}(s_0 \mid s) = \boldsymbol{Q}^{s-s_0} \tag{5.24a}$$

Similarly, (5.20) gives, by iteration,

$$\begin{aligned} P(n, s+1) &= \sum_m P(m, s)\, Q_{mn} \\ &= \sum_{m,k} P(k, s-1)\, Q_{km}\, Q_{mn} = \sum_k P(k, s-1)\left(\boldsymbol{Q}^2\right)_{kn} \\ &\;\;\vdots \\ &= \sum_k P(k, 0)\left(\boldsymbol{Q}^{s+1}\right)_{kn} \end{aligned} \tag{5.25}$$

Setting $\boldsymbol{p}(s)$ to be the column vector with elements $p_n(s) = P(n, s)$, we can write (5.25) as

$$\begin{aligned} \boldsymbol{p}^T(s+1) &= \boldsymbol{p}^T(s)\cdot\boldsymbol{Q} \\ &= \boldsymbol{p}^T(s-1)\cdot\boldsymbol{Q}^2 \\ &\vdots \\ &= \boldsymbol{p}^T(0)\cdot\boldsymbol{Q}^{s+1} \end{aligned} \qquad (5.25a)$$

where $^T$ denotes the transpose operation.

As in quantum mechanics, the Dirac notation greatly clarifies the mathematics.
Reminder: in the Heisenberg picture, the Dirac notation is equivalent to the matrix form.

The state of the system at time $s\,\tau$ is represented by the probability vector $|\,p(s)\rangle$ in the sample space $\mathbb{S}$, which is spanned by the eigenvectors $|\,n\rangle$ of the stochastic variable operator $Y$ with

$$Y\,|\,n\rangle = y(n)\,|\,n\rangle \qquad n = 1, \ldots, M \qquad (a1)$$

Since $\mathbb{S}$ is a real space, we define the inner product of two vectors $\psi$ & $\phi$ in it as

$$\langle\psi\,|\,\phi\rangle = \langle\phi\,|\,\psi\rangle$$

which means the dual vectors are related by a transpose operation

$$\langle\psi\,| = |\,\psi\rangle^T \qquad \forall\ |\,\psi\rangle \in \mathbb{S} \qquad (a2)$$

Using the completeness condition

$$\sum_{m=1}^{M} |\,n\rangle\langle n\,| = I \qquad (a3)$$

we can write any $|\,\psi\rangle \in \mathbb{S}$ as

$$|\,\psi\,\rangle = \sum_{n=1}^{M} |\,n\,\rangle\langle\,n\,|\,\psi\,\rangle = \sum_{n=1}^{M} |\,n\,\rangle\,\psi(n) \qquad (a4)$$

where

$$\psi(n) \equiv \langle n\,|\,\psi\rangle \qquad (a5)$$

Similarly, the inner product of $\psi$ & $\phi$ can be written as

$$\begin{aligned} \langle\psi\,|\,\phi\rangle &= \sum_{n=1}^{M} \langle\psi\,|\,n\rangle\langle n\,|\,\phi\rangle \\ &= \sum_{n=1}^{M} \psi(n)\,\phi(n) \end{aligned} \qquad (a6)$$

which is sometimes used as the definition of the inner product without introducing the concept of dual space.

Probability $P(n, s)$ is then the projection (or component) of $|\,p(s)\rangle$ along $|\,n\rangle$:

$$P(n, s) = \langle p(s)\,|\,n\rangle = \langle n\,|\,p(s)\rangle \qquad (b)$$

From (5.20), we see that the conditional probability $P_{1|1}$ is an operator that relates probability vectors of different times. Hence, we set

$$P_{1|1}(m, s_0\,|\,n, s) = \langle m\,|\,P_{1|1}(s_0\,|\,s)\,|\,n\rangle \qquad (c)$$

so that (5.20) becomes

$$\begin{aligned} \langle p(s+1)\,|\,n\rangle &= \sum_{m=1}^{M} \langle p(s)\,|\,m\rangle\langle m\,|\,P_{1|1}(s\,|\,s+1)\,|\,n\rangle \\ &= \langle p(s)\,|\,P_{1|1}(s\,|\,s+1)\,|\,n\rangle \end{aligned} \qquad (d)$$

Since (d) is true for all $|\,n\,\rangle$, we have

$$\langle p(s+1)\,| = \langle p(s)\,|\,P_{1|1}(s\,|\,s+1) \qquad (e)$$

(5.23) then becomes

$$Q_{mn} = \langle m \mid P_{1|1}(s \mid s+1) \mid n \rangle \qquad \text{(f)}$$

so that $\boldsymbol{Q}$ is just the matrix form of the operator $P_{1|1}(s \mid s+1)$. This justifies the use of $\boldsymbol{Q}$ to denote both the operator $P_{1|1}(s \mid s+1)$ and its matrix representation. (f) thus becomes

$$Q_{mn} = \langle m \mid \boldsymbol{Q} \mid n \rangle \qquad \text{(g)}$$

Similarly, (5.24) becomes

$$\langle m \mid P_{1|1}(s_0 \mid s) \mid n \rangle = (\boldsymbol{Q}^{s-s_0})_{mn}$$

$$\rightarrow \quad P_{1|1}(s_0 \mid s) = P_{1|1}\left(s_0 \mid s_0+1\right) \ldots P_{1|1}(s-1 \mid s) = \boldsymbol{Q}^{s-s_0} \qquad \text{(h)}$$

The eigenvalues of $\boldsymbol{Q}$ are given by the solutions of the secular equation

$$\det \mid \boldsymbol{Q} - \lambda \boldsymbol{I} \mid = 0 \qquad (5.26)$$

where $\boldsymbol{I}$ is the $M \times M$ unit matrix. As can be seen from (f), $\boldsymbol{Q}$ is a real but non-symmetric matrix in the basis $\{ \mid n \rangle \}$. Therefore, the left and right eigenvectors for the same eigenvalue are not equal. Thus,

$$\boldsymbol{Q} \mid \psi_i \rangle = \lambda_i \mid \psi_i \rangle \qquad \text{(i)}$$

$$\langle \chi_i \mid \boldsymbol{Q} = \langle \chi_i \mid \lambda_i \qquad \text{(j)}$$

where $\mid \psi_i \rangle$ & $\langle \chi_i \mid$ are the right & left eigenvectors of $\lambda_i$, respectively, and

$$\mid \chi_i \rangle \neq \mid \psi_i \rangle$$

Taking the transpose of (j) gives

$$\boldsymbol{Q}^T \left| \chi_i \right\rangle = \lambda_i \left| \chi_i \right\rangle \qquad \text{(k)}$$

which is the preferred form when using computer software for calculation.

Caution: for a real but non-symmetric matrix, both its eigenvalues and eigenvectors can be complex. See comment following (5.31).

With the help of the completeness condition and

$$\psi_i(n) = \langle n \mid \psi_i \rangle = \langle \psi_i \mid n \rangle$$

(i) can be written in component form as

$$\sum_n \langle m \mid \boldsymbol{Q} \mid n \rangle \langle n \mid \psi_i \rangle = \lambda_i \langle m \mid \psi_i \rangle$$

$$\rightarrow \quad \sum_n Q_{mn}\, \psi_i(n) = \lambda_i\, \psi_i(m) \qquad (5.28)$$

Similarly, (j) becomes

$$\sum_n \langle \chi_i \mid n \rangle \langle n \mid \boldsymbol{Q} \mid m \rangle = \langle \chi_i \mid m \rangle \lambda_i$$

$$\rightarrow \quad \sum_n \chi_i(n)\, Q_{nm} = \chi_i(m)\, \lambda_i \qquad (5.27)$$

(i) & (j) give

$$\langle \chi_i \mid \boldsymbol{Q} \mid \psi_j \rangle = \lambda_j \langle \chi_i \mid \psi_j \rangle$$

$$\langle \chi_i \mid \boldsymbol{Q} \mid \psi_j \rangle = \lambda_i \langle \chi_i \mid \psi_j \rangle$$

$$\rightarrow \quad 0 = (\lambda_j - \lambda_i) \langle \chi_i \mid \psi_j \rangle$$

$$\therefore \quad \langle \chi_i \mid \psi_j \rangle = 0 \qquad \text{for} \qquad \lambda_i \neq \lambda_j$$

i.e., left and right eigenvectors belonging to different eigenvalues are orthogonal.

For eigenvectors belonging to the same eigenvalue, we can use the Schmidt orthogonalization method to obtain a set of mutually orthogonal vectors.

Therefore, assuming all eigenvectors are properly normalized, we can always write

$$\langle \chi_i \mid \psi_j \rangle = \delta_{ij} \qquad (5.30)$$

$$\rightarrow \quad \sum_{n=1}^{M} \langle \chi_i \mid n \rangle \langle n \mid \psi_j \rangle = \delta_{ij}$$

$$\sum_{n=1}^{M} \chi_i(n)\, \psi_j(n) = \delta_{ij} \qquad (5.30a)$$

Consider the expansion

$$\langle p \mid = \sum_{i=1}^{M} \alpha_i \langle \chi_i \mid \qquad (m)$$

$$\rightarrow \quad \langle p \mid \psi_j \rangle = \sum_{i=1}^{M} \alpha_i \langle \chi_i \mid \psi_j \rangle = \sum_{i=1}^{M} \alpha_i\, \delta_{ij} = \alpha_j$$

Putting this back into (m) gives

$$\langle p \mid = \sum_{i=1}^{M} \langle p \mid \psi_i \rangle \langle \chi_i \mid$$

Since this is true for arbitrary $\langle p \mid$, we must have

$$\sum_{i=1}^{M} \mid \psi_i \rangle \langle \chi_i \mid = I \qquad (5.31)$$

which is the completeness condition for the eigenvectors.

Note that what (5.31) means is that these possibly complex eigenvectors span an $M$-D real space that is also spanned by the $M$ real vectors $\mid n \rangle$. To be consistent, eigenvalues and eigenvectors, if complex, must appear in conjugate pairs. This is so since $\boldsymbol{Q}$ is real.

In terms of components with respect to the basis $\mid n \rangle$, we have

$$\sum_{i=1}^{M} \langle m \mid \psi_i \rangle \langle \chi_i \mid n \rangle = \langle m \mid n \rangle$$

$$\rightarrow \quad \sum_{i=1}^{M} \psi_i(m)\, \chi_i(n) = \delta_{mn} \qquad (5.31a)$$

Taking the absolute value of (5.28) gives

$$\left| \sum_{n=1}^{M} Q_{mn}\, \psi_i(n) \right| = \mid \lambda_i \mid \; \mid \psi_i(m) \mid$$

Since all $Q_{mn}$ are real and non-negative,

$$\mid \lambda_i \mid \; \mid \psi_i(m) \mid \; \leq \sum_n Q_{mn} \mid \psi_i(n) \mid \qquad (5.32)$$

Let

$$\mid \psi_i(n) \mid \; \leq C \qquad \forall\, n$$

then

$$\sum_{n=1}^{M} Q_{mn} \mid \psi_i(n) \mid \leq C \sum_{n=1}^{M} Q_{mn} = C \qquad (5.33)$$

where the discrete version of (5.13)

$$\sum_{n=1}^{M} P_{1|1}(m, s \mid n, s+1) = \sum_{n=1}^{M} Q_{mn} = 1 \qquad (n)$$

was used.

Let

$$\mid \psi_i(n_0) \mid = C$$

then setting $m = n_0$ in (5.32) gives

$$| \lambda_i | \; C \le C$$

$$\to \quad | \lambda_i | \le 1 \qquad \forall \; i \tag{5.34}$$

Using (n), we see that (5.28) can always be satisfied by

$$\lambda_1 = 1 \quad \text{with} \quad \psi_1(n) = c \qquad \forall \; n \tag{o}$$

where $c$ is a normalization constant and we have set arbitrarily $i = 1$.

In other words, every $\boldsymbol{Q}$ has an eigenvalue of 1 with left eigenvector given by (o).

For the right eigenvector, setting $\lambda_1 = 1$ in (5.27) gives

$$\chi_1(m) = \sum_{n=1}^{M} \chi_1(n) \, Q_{n\,m} \tag{5.35}$$

Setting

$$\chi_1(m) = \langle \, p(s) \mid m \, \rangle = P(m, s)$$

(5.35) becomes

$$\begin{aligned} \langle \, p(s) \mid m \, \rangle &= \sum_{n=1}^{M} \langle \, p(s) \mid n \, \rangle \langle \, n \mid P_{1|1}(s \mid s+1) \mid m \, \rangle \\ &= \langle \, p(s+1) \mid m \, \rangle \qquad \text{[ (5.20) used. ]} \end{aligned}$$

$$\to \quad | \, p(s) \, \rangle = | \, p(s+1) \, \rangle$$

$\chi_1$ is therefore a stationary probability vector. We denote this as

$$\chi_1(n) = P_{\text{ST}}(n) \tag{p}$$

The constant $c$ in (o) can be determined from the normalization

$$\langle \, \chi_1 \mid \psi_1 \, \rangle = 1$$

$$\to \quad \sum_{n=1}^{M} \chi_1(n) \, \psi_1(n) = c \sum_{n=1}^{M} \chi_1(n) = 1$$

Using the sum rule for probabilities

$$\sum_{n=1}^{M} \chi_1(n) = \sum_{n=1}^{M} P_{\text{ST}}(n) = 1$$

we have

$$c = 1 \quad \to \quad \psi_1(n) = 1 \qquad \forall \; n \tag{o1}$$

To summarize, every $\boldsymbol{Q}$ has an eigenvalue of 1 whose left eigenvector has every component equals to 1, and right eigenvector that is stationary with the sum of its components equal to 1.

On the other hand, eq($j$)$\times \; | \, \psi_j \rangle$ gives

$$\begin{aligned} \langle \, \chi_i \mid \boldsymbol{Q} \mid \psi_j \, \rangle &= \lambda_i \langle \, \chi_i \mid \psi_j \, \rangle \\ &= \lambda_i \, \delta_{i\,j} \qquad \text{[ (5.30) used. ]} \end{aligned} \tag{q}$$

$$\to \quad \sum_{i=1}^{M} \langle \, \chi_i \mid \boldsymbol{Q} \mid \psi_j \, \rangle = \lambda_j$$

which implies

$$\boldsymbol{Q} = \sum_{i=1}^{M} \lambda_i \mid \psi_i \rangle \langle \chi_i \mid \tag{r}$$

Proof: (r) gives

$$\langle \, \chi_i \mid \boldsymbol{Q} \mid \psi_j \, \rangle = \sum_{k=1}^{M} \lambda_k \langle \, \chi_i \mid \psi_k \, \rangle \langle \, \chi_k \mid \psi_j \, \rangle$$

$$= \sum_{k=1}^{M} \lambda_k \delta_{ik} \delta_{kj} = \lambda_i \delta_{ij}$$

which agrees with (q). QED.

$\langle m \mid \text{eq}(r) \mid n \rangle$ gives

$$Q_{mn} = \sum_{i=1}^{M} \lambda_i \langle m \mid \psi_i \rangle \langle \chi_i \mid n \rangle$$

$$= \sum_{i=1}^{M} \lambda_i \psi_i(m) \chi_i(n) \qquad (5.36)$$

$$= P_{1|1}(m, s \mid n, s+1) \qquad \forall\ s \qquad \text{[ see (5.23). ]}$$

From (5.24), we have

$$P_{1|1}(m, s \mid n, s+2) = (\boldsymbol{Q}^2)_{mn}$$

$$= \sum_{k=1}^{M} Q_{mk} Q_{kn}$$

$$= \sum_{i,j,k=1}^{M} \lambda_i \lambda_j \langle m \mid \psi_i \rangle \langle \chi_i \mid k \rangle \langle k \mid \psi_j \rangle \langle \chi_j \mid n \rangle$$

$$= \sum_{i,j=1}^{M} \lambda_i \lambda_j \langle m \mid \psi_i \rangle \langle \chi_i \mid \psi_j \rangle \langle \chi_j \mid n \rangle$$

$$= \sum_{i,j=1}^{M} \lambda_i \lambda_j \langle m \mid \psi_i \rangle \delta_{ij} \langle \chi_j \mid n \rangle$$

$$= \sum_{i=1}^{M} \lambda_i^2 \langle m \mid \psi_i \rangle \langle \chi_i \mid n \rangle$$

$$= \sum_{i=1}^{M} \lambda_i^2 \psi_i(m) \chi_i(n)$$

Alternatively,

$$(\boldsymbol{Q}^2)_{mn} = \left\langle m \,\middle|\, \boldsymbol{Q}^2 \,\middle|\, n \right\rangle$$

$$= \sum_{i=1}^{M} \left\langle m \,\middle|\, \boldsymbol{Q}^2 \mid \psi_i \right\rangle \langle \chi_i \mid n \rangle$$

$$= \sum_{i=1}^{M} \left\langle m \,\middle|\, \lambda_i^2 \,\middle|\, \psi_i \right\rangle \langle \chi_i \mid n \rangle$$

$$= \sum_{i=1}^{M} \lambda_i^2 \psi_i(m) \chi_i(n)$$

In general,

$$P_{1|1}(m, s_0 \mid n, s) = (\boldsymbol{Q}^{s-s_0})_{mn} = \sum_{i=1}^{M} \lambda_i^{s-s_0} \psi_i(m) \chi_i(n) \qquad (5.37)$$

which, together with the stationary state [see (p)]

$$P_{\text{ST}}(n) = \chi_1(n)$$

provide the general solution to the Markov chain.

$\boldsymbol{Q}$ is called regular if

$$\boldsymbol{Q}^N \neq \boldsymbol{0} \qquad \forall\ N \qquad (5.37a)$$

This means, according to (5.37), any state $n$ can be reached by state $m$ after a time $s - s_0$, provided there are no other prohibitive factors. The necessary condition for a Markov chain to be ergodic is therefore a regular $\boldsymbol{Q}$.

A regular $\boldsymbol{Q}$ can be non-ergodic if it is block-diagonal since states in different blocks cannot be bridged by $\boldsymbol{Q}^N$.

Taking the determinant of (5.37a) gives

$$\det \boldsymbol{Q}^N = (\det \boldsymbol{Q})^N = \left(\prod_i \lambda_i\right)^N = \prod_i \lambda_i^N \neq 0$$

Hence, the necessary and sufficient condition for $\boldsymbol{Q}$ to be regular is

$$\lambda_i \neq 0 \qquad \forall\ i \qquad \text{(5.37b)}$$

Since $|\lambda_i| \leq 1$ with $\lambda_1 = 1$, we write (5.37) as

$$P_{1|1}(m, s_0 \mid n, s) = \psi_1(m)\,\chi_1(n) + \sum_{i=2}^{M} \lambda_i^{s-s_0}\,\psi_i(m)\,\chi_i(n)$$

$$= P_{\text{ST}}(n) + \sum_{i=2}^{M} \lambda_i^{s-s_0}\,\psi_i(m)\,\chi_i(n) \qquad \text{(5.38)}$$

where (o), (o1) & (p) were used.

Now,

$$|\lambda_i| < 1 \qquad \rightarrow \qquad \lim_{s\to\infty} |\lambda_i|^{s-s_0} = 0 \qquad \forall\ i \geq 2$$

$$\therefore \qquad \lim_{s\to\infty} \lambda_i^{s-s_0} = 0 \qquad \forall\ i \geq 2$$

and (5.38) gives

$$\lim_{s\to\infty} P_{1|1}(m, s_0 \mid n, s) = P_{\text{ST}}(n) \qquad \text{(5.39)}$$

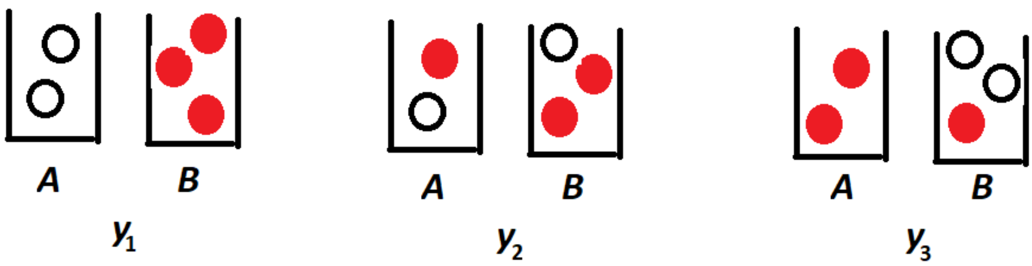
Similarly, (5.25) gives

$$\lim_{s\to\infty} P(n, s) = \sum_{m=1}^{M} P(m, 0)\,P_{\text{ST}}(n)$$

$$= P_{\text{ST}}(n) \qquad [\ (5.6)\ \text{used.}\ ] \qquad \text{(5.40)}$$

## Exercise 5.1.

Consider 2 pots, $A$ & $B$, with 3 red & 2 white balls distributed between them so that $A$ & $B$ always have 2 & 3 balls, respectively. There are 3 different configurations as shown in the figure below.




At each time step, transition between configurations is attained by picking up randomly 1 ball each from $A$ & $B$ and interchanging them.

(a) Compute the transition matrix $\boldsymbol{Q}$ and the conditional probability $P_{1|1}(m, s_0 \mid n, s)$.

(b) If initially $P(1, 0) = 1$, $P(2, 0) = 0$ and $P(3, 0) = 0$, compute the probabilities $P(n, s)$ for $n = 1, 2, 3$ at time $s$.

(c) Assume the realization $y(n) = n$. Compute the first moment, $\langle y(s) \rangle$, and the autocorrelation function, $\langle y(0)\, y(s) \rangle$, for the same initial conditions as (b).

## Answer (a)

Starting with $Y(s) = y_1$, one can only reach $Y(s+1) = y_2$. Therefore

$$Q_{11} = 0 \qquad Q_{12} = 1 \qquad Q_{13} = 0$$

Starting with $Y(s) = y_2$, one can reach $Y(s+1) = y_1$ by exchanging a red ball in $A$ with a white ball in $B$ with a probability $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$. Therefore $Q_{21} = \frac{1}{6}$.

There are two ways to remain in $y_2$, The first is to pick up a white ball from both pots with probability $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$. The second is to pick a red ball from both pots with probability $\frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$ . Therefore, $Q_{22} = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$.

In order to change to $y_3$, one must pick a white ball from $A$ and a red ball from $B$ with probability $\frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$. Therefore $Q_{23} = \frac{1}{3}$.

Starting with $Y(s) = y_3$, there is no way to reach $Y(s+1) = y_1$. Therefore $Q_{31} = 0$.

One can reach $Y(s+1) = y_2$ by exchanging a red ball in $A$ with a white ball in $B$ with a probability $1 \times \frac{2}{3} = \frac{2}{3}$. Therefore $Q_{32} = \frac{2}{3}$.

In order to remain in $y_3$, one must pick a red ball from both $A$ and $B$ with probability $1 \times \frac{1}{3} = \frac{1}{3}$. Therefore $Q_{33} = \frac{1}{3}$.

Thus,

$$\boldsymbol{Q} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix} \qquad \boldsymbol{Q}^2 = \begin{pmatrix} \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{12} & \frac{23}{36} & \frac{5}{18} \\ \frac{1}{9} & \frac{5}{9} & \frac{1}{3} \end{pmatrix}$$

Using the *Mathematica* code in §Code, we get

$$\lambda_1 = 1 \qquad |\psi_1\rangle = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \langle \chi_1 | = \begin{pmatrix} \frac{1}{3} & 2 & 1 \end{pmatrix}$$

$$\lambda_2 = -\frac{1}{3} \qquad |\psi_2\rangle = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \qquad \langle \chi_2 | = \begin{pmatrix} 1 & -2 & 1 \end{pmatrix}$$

$$\lambda_3 = \frac{1}{6} \qquad |\psi_3\rangle = \begin{pmatrix} -\frac{3}{2} \\ -\frac{1}{4} \\ 1 \end{pmatrix} \qquad \langle \chi_3 | = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}$$

The eigenvectors above are mutually orthogonal but not normalized. Keeping $|\psi_i\rangle$ as is, the normalized $\langle \chi_i |$ are easily found to be

$$\langle \chi_1 | = \frac{3}{10} \begin{pmatrix} \frac{1}{3} & 2 & 1 \end{pmatrix} \qquad \langle \chi_2 | = \frac{1}{6} \begin{pmatrix} 1 & -2 & 1 \end{pmatrix} \langle \chi_3 | = \frac{8}{15} \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}$$

The conditional probability matrix is therefore

$$P_{1|1}(s_0 \mid s) = \sum_{i=1}^{3} \lambda_i^{s-s_0} \mid \psi_i \rangle \langle \chi_i \mid$$

$$= \begin{pmatrix} \frac{1}{10} & \frac{3}{5} & \frac{3}{10} \\ \frac{1}{10} & \frac{3}{5} & \frac{3}{10} \\ \frac{1}{10} & \frac{3}{5} & \frac{3}{10} \end{pmatrix} + \left(-\frac{1}{3}\right)^{s-s_0} \begin{pmatrix} \frac{1}{2} & -1 & \frac{1}{2} \\ -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{pmatrix} + \left(\frac{1}{6}\right)^{s-s_0} \begin{pmatrix} \frac{2}{5} & \frac{2}{5} & -\frac{4}{5} \\ \frac{1}{15} & \frac{1}{15} & -\frac{2}{15} \\ -\frac{4}{15} & -\frac{4}{15} & \frac{8}{15} \end{pmatrix}$$

where $\mid \psi_i \rangle \langle \chi_i \mid$ was calculated as “rlProd” in the section Code.

## Answer (b)

The initial condition can be written as

$$\langle p(0) \mid = (1 \quad 0 \quad 0)$$

Using

$$\langle p(s) \mid = \langle p(0) \mid P_{1|1}(0 \mid s)$$

we have [ see Code ]

$$\langle p\,(s) \mid =$$

$$\left(\frac{1}{10} + \frac{1}{2}\left(-\frac{1}{3}\right)^s + \frac{1}{5} \times 2^{1-s} \times 3^{-s},\ \frac{3}{5} + (-1)^{1+s}\,3^{-s} + \frac{1}{5} \times 2^{1-s} \times 3^{-s},\ \frac{3}{10} + \frac{1}{2}\left(-\frac{1}{3}\right)^s - \frac{1}{5} \times 2^{2-s} \times 3^{-s}\right)$$

Note that

$$\begin{aligned} P(n, s) &= \langle p(s) \mid n \rangle \\ &= \sum_m \langle p(0) \mid m \rangle \langle m \mid P_{1|1}(0 \mid s) \mid n \rangle \\ &= \sum_m P(m, 0) \langle m \mid P_{1|1}(0 \mid s) \mid n \rangle \\ &= \langle 1 \mid P_{1|1}(0 \mid s) \mid n \rangle \end{aligned}$$

## Answer (c)

According to (5.7), the 1st moment is

$$\begin{aligned} \langle y(s) \rangle &= \sum_{n=1}^{3} y(n)\, P(n, s) \\ &= \sum_{n=1}^{3} n \,\langle p(s) \mid n \rangle \\ &= \frac{11}{5} - \frac{6^{1-s}}{5} \qquad \text{[ see Code. ]} \end{aligned}$$

For the autocorrelation,

$$\begin{aligned} \langle\, y(0)\, y(s) \rangle &= \sum_{m,n=1}^{3} y(m)\, y(n)\, P_2\,(m, 0\,;\ n, s) \\ &= \sum_{m,n=1}^{3} y(m)\, y(n)\, P\,(m, 0)\, P_{1|1}(m, 0 \mid n, s) \\ &= \frac{11}{5} - \frac{6^{1-s}}{5} \qquad \text{[ see Code. ]} \\ &= \langle y(s) \rangle \end{aligned}$$

This coincidence is due to the special initial conditions & realization we used:

$$\sum_{m,n=1}^{3} y(m)\, y(n)\, P(m, 0)\, P_{1|1}(m, 0 \mid n, s)$$

$$= \sum_{n=1}^{3} y(n)\; P_{1|1}(1, 0 \mid n, s)$$

$$= \sum_{n=1}^{3} n \,\langle 1 \mid P_{1|1}(0 \mid s) \mid n \rangle$$

$$= \sum_{n=1}^{3} n\; P(n, s)$$

$$= \langle y(s) \rangle$$

## Code

```
Q = ( 0    1    0
      1/6  1/2  1/3
      0    2/3  1/3 );

(* eigenvalues & right eigenvectors *)
{λ, rev} = Eigensystem[Q]

{{1, -1/3, 1/6}, {{1, 1, 1}, {3, -1, 1}, {-3/2, -1/4, 1}}}

(* Checking *)
Table[ Q.rev[[i]] == λ[[i]] rev[[i]], {i, 3}]

{True, True, True}

(* eigenvalues & left eigenvectors *)
{λ, lev} = Eigensystem[Q^T]

{{1, -1/3, 1/6}, {{1/3, 2, 1}, {1, -2, 1}, {-1/2, -1/2, 1}}}

(* Checking *)
Table[ lev[[i]].Q == λ[[i]] lev[[i]], {i, 3}]

{True, True, True}

(* normalization constants *)
nconst = MapThread[Dot, {lev, rev}]

{10/3, 6, 15/8}

(* Normalized left eigenvectors; right eigenvectors unchanged *)
lev = lev / nconst

{{1/10, 3/5, 3/10}, {1/6, -1/3, 1/6}, {-4/15, -4/15, 8/15}}
```

```
(* Orthonornality:  ⟨ χi|ψj ⟩=δij  *)
Table[ lev⟦i⟧.rev⟦j⟧, {i, 3}, {j, 3}] // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
(* |ψi⟩⟨ χi| *)
rlProd = Table[Outer[Times, rev⟦i⟧, lev⟦i⟧], {i, 3}];
MatrixForm /@ rlProd
```

$$\left\{\begin{pmatrix} \frac{1}{10} & \frac{3}{5} & \frac{3}{10} \\ \frac{1}{10} & \frac{3}{5} & \frac{3}{10} \\ \frac{1}{10} & \frac{3}{5} & \frac{3}{10} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & -1 & \frac{1}{2} \\ -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{pmatrix}, \begin{pmatrix} \frac{2}{5} & \frac{2}{5} & -\frac{4}{5} \\ \frac{1}{15} & \frac{1}{15} & -\frac{2}{15} \\ -\frac{4}{15} & -\frac{4}{15} & \frac{8}{15} \end{pmatrix}\right\}$$

```
(* Completeness:  ∑_{i = 1}^{3} |ψi⟩⟨ χi| = I  *)
```

$$\sum_{i=1}^{3} \text{rlProd}[\![i]\!] \; // \; \text{MatrixForm}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
(* P1|1(s0|s) *)
P11[s0_, s_] := Sum[ev⟦i⟧^(s-s0) rlProd⟦i⟧, {i, 3}]

P11[0, s] // MatrixForm
```

$$\begin{pmatrix} \frac{1}{10}+\frac{1}{2}\left(-\frac{1}{3}\right)^{s}+\frac{1}{5}\times 2^{1-s}\times 3^{-s} & \frac{3}{5}+(-1)^{1+s}\,3^{-s}+\frac{1}{5}\times 2^{1-s}\times 3^{-s} & \frac{3}{10}+\frac{1}{2}\left(-\frac{1}{3}\right)^{s}-\frac{1}{5}\times 2^{2-s}\times 3^{-s} \\ \frac{1}{10}-\frac{1}{2}(-1)^{s}\,3^{-1-s}+\frac{1}{5}\times 2^{-s}\times 3^{-1-s} & \frac{3}{5}+(-1)^{s}\,3^{-1-s}+\frac{1}{5}\times 2^{-s}\times 3^{-1-s} & \frac{3}{10}-\frac{1}{2}(-1)^{s}\,3^{-1-s}-\frac{1}{5}\times 2^{1-s}\times 3^{-} \\ \frac{1}{10}+\frac{1}{2}(-1)^{s}\,3^{-1-s}-\frac{1}{5}\times 2^{2-s}\times 3^{-1-s} & \frac{3}{5}+\left(-\frac{1}{3}\right)^{1+s}-\frac{1}{5}\times 2^{2-s}\times 3^{-1-s} & \frac{3}{10}+\frac{1}{2}(-1)^{s}\,3^{-1-s}+\frac{1}{5}\times 2^{3-s}\times 3^{-} \end{pmatrix}$$

```
(* Q *)
P11[s, s + 1] // MatrixForm
```

$$\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

```
(* Q^2 *)
P11[s, s + 2] // MatrixForm
```

$$\begin{pmatrix} \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{12} & \frac{23}{36} & \frac{5}{18} \\ \frac{1}{9} & \frac{5}{9} & \frac{1}{3} \end{pmatrix}$$

```
(* Answer to (b) *)
p0 = {1, 0, 0};
ps = p0 . P11[0, s]
```

$$\left\{\frac{1}{10}+\frac{1}{2}\left(-\frac{1}{3}\right)^{s}+\frac{1}{5}\times 2^{1-s}\times 3^{-s},\ \frac{3}{5}+(-1)^{1+s}\,3^{-s}+\frac{1}{5}\times 2^{1-s}\times 3^{-s},\ \frac{3}{10}+\frac{1}{2}\left(-\frac{1}{3}\right)^{s}-\frac{1}{5}\times 2^{2-s}\times 3^{-s}\right\}$$

```
(* Answer to (c) *)
```

```
(* 1st moment *)
```

$$yb = \sum_{n=1}^{3} n\, ps[[n]] \text{ // Simplify}$$

$$\frac{11}{5} - \frac{6^{1-s}}{5}$$

```
(* autocorrelation function *)
```

$$yyb = \sum_{m=1}^{3} \sum_{n=1}^{3} m\, n\, p0[[m]]\, P11[0, s][[m, n]] \text{ // Simplify}$$

$$\frac{11}{5} - \frac{6^{1-s}}{5}$$