5.C.1. Spectral Properties

For a time independent $\boldsymbol{Q}$, we have

$$Q_{mn}(s) = P_{1|1}(m, s \mid n, s+1) = P_{1|1}(m, 0 \mid n, 1) = Q_{mn} \qquad \forall\, s \qquad (5.23)$$

(5.21) can be solved by iteration starting with

$$P_{1|1}(m, s \mid n, s) = \delta_{mn}$$

For the 1st time step (or iteration), (5.21) becomes

$$\begin{aligned} P_{1|1}(m, s \mid n, s+1) &= \sum_k P_{1|1}(m, s \mid k, s)\, P_{1|1}(k, s \mid n, s+1) \\ &= \sum_k \delta_{mk}\, P_{1|1}(k, s \mid n, s+1) \\ &= P_{1|1}(m, s \mid n, s+1) \\ &= Q_{mn} \end{aligned}$$

For the 2nd time step,

$$\begin{aligned} P_{1|1}(m, s-1 \mid n, s+1) &= \sum_k P_{1|1}(m, s-1 \mid k, s)\, P_{1|1}(k, s \mid n, s+1) \\ &= \sum_k P_{1|1}(m, s-1 \mid k, s)\, P_{1|1}(k, s \mid n, s+1) \\ &= \sum_k Q_{mk}\, Q_{kn} \\ &= (\boldsymbol{Q}^2)_{mn} \end{aligned}$$

For the 3rd time step,

$$\begin{aligned} P_{1|1}(m, s-2 \mid n, s+1) &= \sum_k P_{1|1}(m, s-2 \mid k, s)\, P_{1|1}(k, s \mid n, s+1) \\ &= \sum_k (\boldsymbol{Q}^2)_{mk}\, Q_{kn} \\ &= (\boldsymbol{Q}^3)_{mn} \end{aligned}$$

In general,

$$P_{1|1}\left(m, s_0 \mid n, s\right) = (\boldsymbol{Q}^{s-s_0})_{mn} \qquad (5.24)$$

Similarly, (5.20) gives, by iteration,

$$\begin{aligned} P(n, s+1) &= \sum_m P(m, s)\, Q_{mn} \\ &= \sum_{m,k} P(k, s-1)\, Q_{km}\, Q_{mn} = \sum_k P(k, s-1)\, (\boldsymbol{Q}^2)_{kn} \\ &\;\;\vdots \\ &= \sum_k P(k, 0)\, (\boldsymbol{Q}^{s+1})_{kn} \end{aligned} \qquad (5.25)$$

As in quantum mechanics, the Dirac notation greatly clarifies the mathematics.

The state of the system at time $s\,\tau$ is represented by the probability vector $\mid p(s)\rangle$ in the sample space, which is spanned by the eigenvectors $\mid n\rangle$ of the stochastic variable operator Y with

$$Y \mid n\rangle = y(n) \mid n\rangle \qquad n = 1, \ldots, M$$

$$\langle n \mid = \mid n\rangle^T \qquad {}^T = \text{transpose operator}$$

Note: $\mid p(s)\rangle$ is analogous to the state vector $\mid \psi(t)\,\rangle$ in the Hilbert space of quantum mechanics except that it is real.

Probability $P(n, s)$ is then the projection (or component) of $\mid p(s)\rangle$ along $\mid n\rangle$:

$$P(n, s) = \langle p(s) \mid n \rangle = \langle n \mid p(s) \rangle \quad \text{(a)}$$

From (5.20), we see that the conditional probability $P_{1|1}$ is an operator that relates probability vectors of different times. Hence, we set

$$P_{1|1}(m, s_0 \mid n, s) = \langle m \mid P_{1|1}(s_0 \mid s) \mid n \rangle \quad \text{(b)}$$

so that (5.20) becomes

$$\langle p(s+1) \mid n \rangle = \sum_{m=1}^{M} \langle p(s) \mid m \rangle \langle m \mid P_{1|1}(s \mid s+1) \mid n \rangle$$
$$= \langle p(s) \mid P_{1|1}(s \mid s+1) \mid n \rangle \quad \text{(c)}$$

where the completeness condition

$$\sum_{m=1}^{M} \mid m \rangle \langle m \mid = I \quad \text{(d)}$$

was used.

Since (5.25a) is true for all $\mid n \rangle$, we have

$$\langle p(s+1) \mid = \langle p(s) \mid P_{1|1}(s \mid s+1) \quad \text{(e)}$$

(5.23) then becomes

$$Q_{m\,n} = \langle m \mid P_{1|1}(s \mid s+1) \mid n \rangle \quad \text{(f)}$$

so that $\boldsymbol{Q}$ is just the matrix form of the operator $P_{1|1}(s \mid s+1)$. This justifies the use of $\boldsymbol{Q}$ to denote both the operator $P_{1|1}(s \mid s+1)$ and its matrix representation. (f) thus becomes

$$Q_{m\,n} = \langle m \mid \boldsymbol{Q} \mid n \rangle \quad \text{(g)}$$

Similarly, (5.24) becomes

$$\langle m \mid P_{1|1}(s_0 \mid s) \mid n \rangle = (\boldsymbol{Q}^{s-s_0})_{m\,n}$$
$$\rightarrow \quad P_{1|1}(s_0 \mid s) = P_{1|1}(s_0 \mid s_0+1) \ldots P_{1|1}(s-1 \mid s) = \boldsymbol{Q}^{s-s_0} \quad \text{(h)}$$

The eigenvalues of $\boldsymbol{Q}$ are given by the solutions of the secular equation

$$\det \mid \boldsymbol{Q} - \lambda \boldsymbol{I} \mid = 0 \quad (5.26)$$

where $\boldsymbol{I}$ is the $M \times M$ unit matrix. As can be seen from (f) & (h), $\boldsymbol{Q}$ is a real but non-symmetric matrix in the basis $\{ \mid n \rangle \}$. Therefore, the left and right eigenvectors for the same eigenvalue are not equal. Let

$$\boldsymbol{Q} \mid \psi_i \rangle = \lambda_i \mid \psi_i \rangle \quad \text{(i)}$$
$$\langle \chi_i \mid \boldsymbol{Q} = \langle \chi_i \mid \lambda_i \quad \text{(j)}$$

Since $\boldsymbol{Q}$ is always real, we define

$$\langle \chi_i \mid = \mid \chi_i \rangle^T$$

(j) is then equivalent to

$$\boldsymbol{Q}^T \mid \chi_i \rangle = \lambda_i \mid \chi_i \rangle \quad \text{(k)}$$

which is the preferred form when using computer software for calculation.

Caution: for a real but non-symmetric matrix, both its eigenvalues and eigenvectors can be complex.

With the help of the completeness condition and

$$\psi_i(n) = \langle n \mid \psi_i \rangle = \langle \psi_i \mid n \rangle$$

(i) can be written as

$$\sum_n \langle m \mid \boldsymbol{Q} \mid n \rangle \langle n \mid \psi_i \rangle = \lambda_i \langle m \mid \psi_i \rangle$$
$$\rightarrow \quad \sum_n Q_{m\,n}\, \psi_i(n) = \lambda_i\, \psi_i(m) \quad (5.28)$$

Similarly, (k) becomes

$$\sum_n \langle \chi_i \mid n \rangle \langle n \mid \boldsymbol{Q} \mid m \rangle = \langle \chi_i \mid m \rangle \lambda_i$$

$$\rightarrow \quad \sum_n \chi_i(n)\, Q_{n\,m} = \chi_i(m)\, \lambda_i \tag{5.27}$$

(i) & (j) give

$$\langle \chi_i \mid \boldsymbol{Q} \mid \psi_j \rangle = \lambda_j \langle \chi_i \mid \psi_j \rangle$$

$$\langle \chi_i \mid \boldsymbol{Q} \mid \psi_j \rangle = \lambda_i \langle \chi_i \mid \psi_j \rangle$$

$$\rightarrow \quad 0 = (\lambda_j - \lambda_i) \langle \chi_i \mid \psi_j \rangle$$

$$\therefore \quad \langle \chi_i \mid \psi_j \rangle = 0 \qquad \text{for} \qquad \lambda_i \neq \lambda_j$$

i.e., left and right eigenvectors belonging to different eigenvalues are orthogonal.

For eigenvectors belonging to the same eigenvalue, we can use the Schmidt orthogonalization method to obtain a set of mutually orthogonal vectors.

Therefore, assuming all eigenvectors are properly normalized, we can always write

$$\langle \chi_i \mid \psi_j \rangle = \delta_{i\,j} \tag{5.30}$$

$$\rightarrow \quad \sum_{n=1}^{M} \langle \chi_i \mid n \rangle \langle n \mid \psi_j \rangle = \delta_{i\,j}$$

$$\sum_{n=1}^{M} \chi_i(n)\, \psi_j(n) = \delta_{i\,j} \tag{5.30a}$$

Consider the expansion

$$\langle p \mid = \sum_{i=1}^{M} \alpha_i \langle \chi_i \mid \tag{m}$$

$$\rightarrow \quad \langle p \mid \psi_j \rangle = \sum_{i=1}^{M} \alpha_i \langle \chi_i \mid \psi_j \rangle = \sum_{i=1}^{M} \alpha_i\ \delta_{i\,j} = \alpha_j$$

Putting this back into (m) gives

$$\langle p \mid = \sum_{i=1}^{M} \langle p \mid \psi_i \rangle \langle \chi_i \mid$$

Since this is true for arbitrary $\langle p \mid$, we must have

$$\sum_{i=1}^{M} \mid \psi_i \rangle \langle \chi_i \mid = \boldsymbol{I} \tag{5.31}$$

which is the completeness condition for the eigenvectors.

Note that what (5.31) means is that these possibly complex eigenvectors span an M-D real space that is also spanned by the M real vectors $\mid n \rangle$. To be consistent, eigenvalues and eigenvectors, if complex, must appear in conjugate pairs. This is so since $\boldsymbol{Q}$ is real.

In terms of components with respect to the basis $\mid n \rangle$, we have

$$\sum_{i=1}^{M} \langle m \mid \psi_i \rangle \langle \chi_i \mid n \rangle = \langle m \mid n \rangle$$

$$\rightarrow \quad \sum_{i=1}^{M} \psi_i(m)\, \chi_i(n) = \delta_{m\,n} \tag{5.31a}$$

Taking the absolute value of (5.28) gives

$$\left| \sum_{n=1}^{M} Q_{m\,n}\, \psi_i(n) \right| = |\,\lambda_i\,| \;\; |\,\psi_i(m)\,|$$

Since all $Q_{m\,n}$ are real,

$$|\,\lambda_i\,| \;\; |\,\psi_i(m)\,| \;\le \sum_{n} Q_{m\,n}\, |\,\psi_i(n)\,| \tag{5.32}$$

Let

$$|\,\psi_i(n)\,| \;\le C \qquad\qquad \forall\; n$$

then

$$\sum_{n=1}^{M} Q_{m\,n}\, |\,\psi_i(n)\,| \le C \sum_{n=1}^{M} Q_{m\,n} \; = C \tag{5.33}$$

where the discrete version of (5.13)

$$\sum_{n=1}^{M} P_{1|1}(m,\,s \mid n,\,s+1) = \sum_{n=1}^{M} Q_{m\,n} = 1 \tag{n}$$

was used.

Let

$$|\,\psi_i\,(n_0)\,| = C$$

then setting $m = n_0$ in (5.32) gives

$$|\,\lambda_i\,| \;\; C \le C$$

$$\rightarrow \qquad |\,\lambda_i\,| \;\le 1 \qquad\qquad \forall\; i \tag{5.34}$$

Using (n), we see that (5.28) can always be satisfied by

$$\lambda_1 = 1 \quad \text{with} \qquad \psi_1(n) = c \qquad\quad \forall\; n \tag{o}$$

where c is a normalization constant and we have set arbitrarily $i = 1$.

In other words, every $\boldsymbol{Q}$ has an eigenvalue of 1 with left eigenvector given by (o).

For the right eigenvector, setting $\lambda_1 = 1$ in (5.27) gives

$$\chi_1(m) = \sum_{n=1}^{M} \chi_1(n)\, Q_{n\,m} \tag{5.35}$$

Setting

$$\chi_1(m) = \langle\, p(s) \mid m\,\rangle = P(m,\,s)$$

(5.35) becomes

$$\langle\, p(s) \mid m\,\rangle = \sum_{n=1}^{M} \langle\, p(s) \mid n\,\rangle \langle\, n \mid P_{1|1}(s \mid s+1) \mid m\,\rangle$$

$$= \langle\, p(s+1) \mid m\,\rangle \qquad\qquad [\text{ (5.20) used. }]$$

$$\rightarrow \qquad |\, p(s)\,\rangle = |\, p(s+1)\,\rangle$$

χ_1 is therefore a stationary probability vector. We denote this as

$$\chi_1(n) = P_{\text{ST}}(n) \tag{p}$$

The constant c in (o) can be determined from the normalization

$$\langle\, \chi_1 \mid \psi_1\,\rangle = 1$$

$$\rightarrow \qquad \sum_{n=1}^{M} \chi_1(n)\, \psi_1(n) = c \sum_{n=1}^{M} \chi_1(n) = 1$$

Using the sum rule for probabilities

$$\sum_{n=1}^{M} \chi_1(n) = \sum_{n=1}^{M} P_{\text{ST}}(n) = 1$$

we have

$$c = 1 \quad \to \quad \psi_1(n) = 1 \qquad \forall\ n \qquad \text{(o')}$$

On the other hand, $(l)\ |\ \psi_j\rangle$ gives

$$\langle \chi_i \mid \boldsymbol{Q} \mid \psi_j \rangle = \lambda_i \langle \chi_i \mid \psi_j \rangle$$

$$= \lambda_i\, \delta_{ij} \qquad \text{[(5.30) used.]} \qquad \text{(q)}$$

$$\to \quad \sum_{i=1}^{M} \langle \chi_i \mid \boldsymbol{Q} \mid \psi_j \rangle = \lambda_j$$

which implies

$$\boldsymbol{Q} = \sum_{i=1}^{M} \lambda_i \mid \psi_i \rangle \langle \chi_i \mid \qquad \text{(r)}$$

Proof: (r) gives

$$\langle \chi_i \mid \boldsymbol{Q} \mid \psi_j \rangle = \sum_{k=1}^{M} \lambda_k \langle \chi_i \mid \psi_k \rangle \langle \chi_k \mid \psi_j \rangle$$

$$= \sum_{k=1}^{M} \lambda_k\, \delta_{ik}\, \delta_{kj} = \lambda_i\, \delta_{ij}$$

which is (p).

$\langle m \mid (q) \mid n \rangle$ gives

$$Q_{mn} = \sum_{i=1}^{M} \lambda_i \langle m \mid \psi_i \rangle \langle \chi_i \mid n \rangle$$

$$= \sum_{i=1}^{M} \lambda_i\, \psi_i(m)\, \chi_i(n) \qquad \text{(5.36)}$$

$$= P_{1|1}(m, s \mid n, s+1) \qquad \forall\ s \qquad \text{[see (5.23).]}$$

From (5.24), we have

$$P_{1|1}(m, s \mid n, s+2) = \left(\boldsymbol{Q}^2\right)_{mn}$$

$$= \sum_{i,j=1}^{M} \lambda_i \lambda_j \langle m \mid \psi_i \rangle \langle \chi_i \mid \psi_j \rangle \langle \chi_j \mid n \rangle$$

$$= \sum_{i=1}^{M} \lambda_i^2 \langle m \mid \psi_i \rangle \langle \chi_i \mid n \rangle$$

$$= \sum_{i=1}^{M} \lambda_i^2\, \psi_i(m)\, \chi_i(n)$$

By induction, we have

$$P_{1|1}(m, s_0 \mid n, s) = \left(\boldsymbol{Q}^{s-s_0}\right)_{mn} = \sum_{i=1}^{M} \lambda_i^{s-s_0}\, \psi_i(m)\, \chi_i(n) \qquad \text{(5.37)}$$

which, together with

$$P(n, s) = \chi_i(n)$$

provide the general solution to the Markov chain.

$\boldsymbol{Q}$ is called regular if

$$\boldsymbol{Q}^N \neq \boldsymbol{0} \qquad \forall\ N$$

Since $|\ \lambda_i\ | \leq 1$ with $\lambda_1 = 1$, we write (5.37) as

$$P_{1|1}(m, s_0 \mid n, s) = \psi_1(m)\, \chi_1(n) + \sum_{i=2}^{M} \lambda_i^{s-s_0}\, \psi_i(m)\, \chi_i(n)$$

$$= P_{ST}(n) + \sum_{i=2}^{M} \lambda_i^{s-s_0} \psi_i(m)\, \chi_i(n) \tag{5.38}$$

where (o), (o') & (p) were used.

Now,

$$|\lambda_i| < 1 \quad \rightarrow \quad \lim_{s\to\infty} |\lambda_i|^{s-s_0} = 0 \qquad \forall\, i \geq 2$$

$$\therefore \quad \lim_{s\to\infty} \lambda_i^{s-s_0} = 0 \qquad \forall\, i \geq 2$$

and (5.38) gives

$$\lim_{s\to\infty} P_{1|1}(m, s_0 \mid n, s) = P_{ST}(n) \tag{5.39}$$

Similarly, (5.25) gives

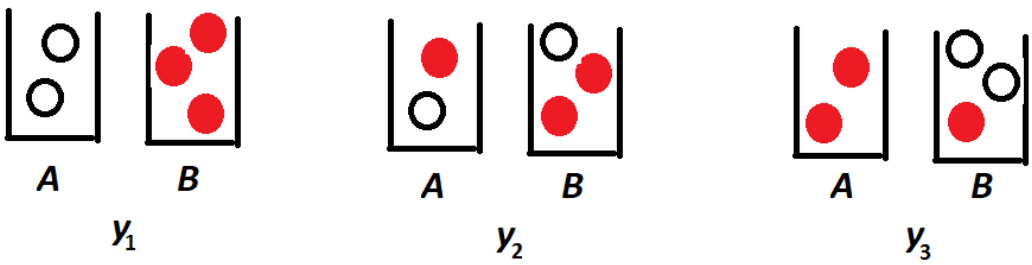
$$\lim_{s\to\infty} P(n, s) = \sum_{m=1}^{M} P(m, 0)\, P_{ST}(n)$$

$$= P_{ST}(n) \qquad [\text{ (5.6) used. }] \tag{5.40}$$

Markov chains with regular $\boldsymbol{Q}$ are ergodic, which means every state is eventually reachable starting from any state. This is no longer true if $\boldsymbol{Q}$ is block diagonal. In which case, multiple long-time states exist.

Exercise 5.1.

Consider 2 pots, A & B, with 3 red & 2 white balls distributed between them so that A always has 2 & B has 3 balls. There are 3 different configurations as shown in the figure below.



At each time step, transition between configurations is attained by picking up randomly 1 ball each from A & B and interchanging them.

(a) Compute the transition matrix $\boldsymbol{Q}$ and the conditional probability $P_{1|1}(m, s_0 \mid n, s)$.

(b) If initially $P(1, 0) = 1$, $P(2, 0) = 0$ and $P(3, 0) = 0$, compute the probabilities $P(n, s)$ for $n = 1, 2, 3$ at time s.

(c) Assume the realization $y(n) = n$. Compute the first moment, $\langle y(s) \rangle$, and the autocorrelation function, $\langle y(0)\, y(s) \rangle$, for the same initial conditions as (b).

Answer (a)

Starting with $Y(s) = y_1$, one can only reach $Y(s+1) = y_2$. Therefore

$$Q_{11} = 0 \; Q_{12} = 1 \; Q_{13} = 0$$

Starting with $Y(s) = y_2$, one can reach $Y(s+1) = y_1$ by exchanging a red ball in A with a white ball in B with a probability $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$. Therefore $Q_{21} = \frac{1}{6}$.

There are two ways to remain in y_2, The first is to pick up a white ball from both pots with probability $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$. The second is to pick a red ball from both pots with probability $\frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$. Therefore, $Q_{22} = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$.

In order to change to y_3, one must pick a white ball from A and a red ball from B with probability $\frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$. Therefore $Q_{23} = \frac{1}{3}$.

Starting with $Y(s) = y_3$, there is no way to reach $Y(s+1) = y_1$. Therefore $Q_{31} = 0$.

One can reach $Y(s+1) = y_2$ by exchanging a red ball in A with a white ball in B with a probability $1 \times \frac{2}{3} = \frac{2}{3}$. Therefore $Q_{32} = \frac{2}{3}$.

In order to remain in y_3, one must pick a red ball from both A and B with probability $1 \times \frac{1}{3} = \frac{1}{3}$. Therefore $Q_{33} = \frac{1}{3}$.

Thus,

$$\boldsymbol{Q} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix} \qquad \boldsymbol{Q}^2 = \begin{pmatrix} \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{12} & \frac{23}{36} & \frac{5}{18} \\ \frac{1}{9} & \frac{5}{9} & \frac{1}{3} \end{pmatrix}$$

Using the *Mathematica* code in the section Code, we get

$$\lambda_1 = 1 \qquad |\,\psi_1\rangle = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \langle\,\chi_1\,| = \left(\tfrac{1}{3}\ \ 2\ \ 1\right)$$

$$\lambda_2 = -\frac{1}{3} \qquad |\,\psi_2\rangle = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \qquad \langle\,\chi_2\,| = (1\ \ -2\ \ 1)$$

$$\lambda_3 = \frac{1}{6} \qquad |\,\psi_3\rangle = \begin{pmatrix} -\frac{3}{2} \\ -\frac{1}{4} \\ 1 \end{pmatrix} \qquad \langle\,\chi_3\,| = \left(-\tfrac{1}{2}\ \ -\tfrac{1}{2}\ \ 1\right)$$

The eigenvectors above are mutually orthogonal not normalized. Keeping $|\,\psi_i\rangle$ as is, the normalized $\langle\chi_i\,|$ are easily found to be

$$\langle\,\chi_1\,| = \frac{3}{10}\left(\tfrac{1}{3}\ \ 2\ \ 1\right) \langle\,\chi_2\,| = \frac{1}{6}(1\ \ -2\ \ 1) \langle\,\chi_3\,| = \frac{8}{15}\left(-\tfrac{1}{2}\ \ -\tfrac{1}{2}\ \ 1\right)$$

The conditional probability matrix is therefore

$$P_{1|1}(s_0\,|\,s) = \sum_{i=1}^{3} \lambda_i^{s-s_0}\,|\,\psi_i\rangle\langle\chi_i\,|$$
$$= \begin{pmatrix} \frac{1}{10} & \frac{3}{5} & \frac{3}{10} \\ \frac{1}{10} & \frac{3}{5} & \frac{3}{10} \\ \frac{1}{10} & \frac{3}{5} & \frac{3}{10} \end{pmatrix} + \left(-\frac{1}{3}\right)^{s-s_0} \begin{pmatrix} \frac{1}{2} & -1 & \frac{1}{2} \\ -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{pmatrix} + \left(\frac{1}{6}\right)^{s-s_0} \begin{pmatrix} \frac{2}{5} & \frac{2}{5} & -\frac{4}{5} \\ \frac{1}{15} & \frac{1}{15} & -\frac{2}{15} \\ -\frac{4}{15} & -\frac{4}{15} & \frac{8}{15} \end{pmatrix}$$

where $|\,\psi_i\rangle\langle\chi_i\,|$ was calculated as “rlProd” in the section Code.

Answer (b)

The initial condition can be written as

$$\langle p(0) \mid = (1 \quad 0 \quad 0)$$

Using

$$\langle p(s) \mid = \langle p(0) \mid P_{1|1}(0 \mid s)$$

we have [see Code]

$$\langle p(s) \mid =$$

$$\left(\frac{1}{10} + \frac{1}{2}\left(-\frac{1}{3}\right)^s + \frac{1}{5} \times 2^{1-s} \times 3^{-s},\ \frac{3}{5} + (-1)^{1+s}\, 3^{-s} + \frac{1}{5} \times 2^{1-s} \times 3^{-s},\ \frac{3}{10} + \frac{1}{2}\left(-\frac{1}{3}\right)^s - \frac{1}{5} \times 2^{2-s} \times 3^{-s} \right)$$

Note that

$$\begin{aligned} P(n, s) &= \langle p(s) \mid n \rangle \\ &= \sum_m \langle p(0) \mid m \rangle \langle m \mid P_{1|1}(0 \mid s) \mid n \rangle \\ &= \sum_m P(m, 0) \langle m \mid P_{1|1}(0 \mid s) \mid n \rangle \\ &= \langle 1 \mid P_{1|1}(0 \mid s) \mid n \rangle \end{aligned}$$

Answer (c)

According to (5.7), the 1st moment is

$$\begin{aligned} \langle y(s) \rangle &= \sum_{n=1}^{3} y(n)\, P(n, s) \\ &= \sum_{n=1}^{3} n \,\langle p(s) \mid n \rangle \\ &= \frac{11}{5} - \frac{6^{1-s}}{5} \qquad \text{[see Code.]} \end{aligned}$$

For the autocorrelation,

$$\begin{aligned} \langle\, y(0)\, y(s) \rangle &= \sum_{m,n=1}^{3} y(m)\, y(n)\, P_2\,(m, 0\,;\ n, s) \\ &= \sum_{m,n=1}^{3} y(m)\, y(n)\, P\,(m, 0)\, P_{1|1}(m, 0 \mid n, s) \\ &= \frac{11}{5} - \frac{6^{1-s}}{5} \qquad \text{[see Code.]} \\ &= \langle y(s) \rangle \end{aligned}$$

This coincidence is due to the special initial conditions & realization we used:

$$\begin{aligned} &\sum_{m,n=1}^{3} y(m)\, y(n)\, P\,(m, 0)\, P_{1|1}(m, 0 \mid n, s) \\ &= \sum_{n=1}^{3} y(n)\ P_{1|1}(1, 0 \mid n, s) \\ &= \sum_{n=1}^{3} n \,\langle 1 \mid P_{1|1}(0 \mid s) \mid n \rangle \\ &= \sum_{n=1}^{3} n\ P(n, s) \\ &= \langle y(s) \rangle \end{aligned}$$

Code

```
Q = (0    1    0
     1/6  1/2  1/3
     0    2/3  1/3);
```

```
(* eigenvalues & right eigenvectors *)
{λ, rev} = Eigensystem[Q]
```

$\left\{\left\{1, -\frac{1}{3}, \frac{1}{6}\right\}, \left\{\{1, 1, 1\}, \{3, -1, 1\}, \left\{-\frac{3}{2}, -\frac{1}{4}, 1\right\}\right\}\right\}$

```
(* Checking *)
Table[ Q.rev[[i]] == λ[[i]] rev[[i]], {i, 3}]
```

{True, True, True}

```
(* eigenvalues & left eigenvectors *)
{λ, lev} = Eigensystem[Q^T]
```

$\left\{\left\{1, -\frac{1}{3}, \frac{1}{6}\right\}, \left\{\left\{\frac{1}{3}, 2, 1\right\}, \{1, -2, 1\}, \left\{-\frac{1}{2}, -\frac{1}{2}, 1\right\}\right\}\right\}$

```
(* Checking *)
Table[ lev[[i]].Q == λ[[i]] lev[[i]], {i, 3}]
```

{True, True, True}

```
(* normalization constants *)
nconst = MapThread[Dot, {lev, rev}]
```

$\left\{\frac{10}{3}, 6, \frac{15}{8}\right\}$

```
(* Normalized left eigenvectors; right eigenvectors unchanged *)
lev = lev / nconst
```

$\left\{\left\{\frac{1}{10}, \frac{3}{5}, \frac{3}{10}\right\}, \left\{\frac{1}{6}, -\frac{1}{3}, \frac{1}{6}\right\}, \left\{-\frac{4}{15}, -\frac{4}{15}, \frac{8}{15}\right\}\right\}$

```
(* Orthonornality:  ⟨ χi|ψj ⟩=δij  *)
Table[ lev[[i]].rev[[j]], {i, 3}, {j, 3}] // MatrixForm
```

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

```
(* |ψi⟩⟨ χi| *)
rlProd = Table[Outer[Times, rev[[i]], lev[[i]]], {i, 3}];
MatrixForm /@ rlProd
```

$\left\{\begin{pmatrix} \frac{1}{10} & \frac{3}{5} & \frac{3}{10} \\ \frac{1}{10} & \frac{3}{5} & \frac{3}{10} \\ \frac{1}{10} & \frac{3}{5} & \frac{3}{10} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & -1 & \frac{1}{2} \\ -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{pmatrix}, \begin{pmatrix} \frac{2}{5} & \frac{2}{5} & -\frac{4}{5} \\ \frac{1}{15} & \frac{1}{15} & -\frac{2}{15} \\ -\frac{4}{15} & -\frac{4}{15} & \frac{8}{15} \end{pmatrix}\right\}$

(* Completeness: $\sum_{i=1}^{3} |\psi_i\rangle\langle\chi_i| = I$ *)

$\sum_{i=1}^{3}$ rlProd[[i]] // MatrixForm

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(* $P_{1|1}(s0|s)$ *)

P11[s0_, s_] := Sum[ev[[i]]$^{s-s0}$ rlProd[[i]], {i, 3}]

P11[0, s] // MatrixForm

$$\begin{pmatrix} \frac{1}{10}+\frac{1}{2}\left(-\frac{1}{3}\right)^{s}+\frac{1}{5}\times 2^{1-s}\times 3^{-s} & \frac{3}{5}+(-1)^{1+s}\,3^{-s}+\frac{1}{5}\times 2^{1-s}\times 3^{-s} & \frac{3}{10}+\frac{1}{2}\left(-\frac{1}{3}\right)^{s}-\frac{1}{5}\times 2^{2-s}\times 3^{-s} \\ \frac{1}{10}-\frac{1}{2}(-1)^{s}\,3^{-1-s}+\frac{1}{5}\times 2^{-s}\times 3^{-1-s} & \frac{3}{5}+(-1)^{s}\,3^{-1-s}+\frac{1}{5}\times 2^{-s}\times 3^{-1-s} & \frac{3}{10}-\frac{1}{2}(-1)^{s}\,3^{-1-s}-\frac{1}{5}\times 2^{1-s}\times 3^{-} \\ \frac{1}{10}+\frac{1}{2}(-1)^{s}\,3^{-1-s}-\frac{1}{5}\times 2^{2-s}\times 3^{-1-s} & \frac{3}{5}+\left(-\frac{1}{3}\right)^{1+s}-\frac{1}{5}\times 2^{2-s}\times 3^{-1-s} & \frac{3}{10}+\frac{1}{2}(-1)^{s}\,3^{-1-s}+\frac{1}{5}\times 2^{3-s}\times 3^{-} \end{pmatrix}$$

(* Q *)

P11[s, s + 1] // MatrixForm

$$\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

(* Q^2 *)

P11[s, s + 2] // MatrixForm

$$\begin{pmatrix} \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{12} & \frac{23}{36} & \frac{5}{18} \\ \frac{1}{9} & \frac{5}{9} & \frac{1}{3} \end{pmatrix}$$

(* Answer to (b) *)

p0 = {1, 0, 0};

ps = p0 . P11[0, s]

$$\left\{\frac{1}{10}+\frac{1}{2}\left(-\frac{1}{3}\right)^{s}+\frac{1}{5}\times 2^{1-s}\times 3^{-s},\ \frac{3}{5}+(-1)^{1+s}\,3^{-s}+\frac{1}{5}\times 2^{1-s}\times 3^{-s},\ \frac{3}{10}+\frac{1}{2}\left(-\frac{1}{3}\right)^{s}-\frac{1}{5}\times 2^{2-s}\times 3^{-s}\right\}$$

(* Answer to (c) *)

(* 1st moment *)

yb = $\sum_{n=1}^{3}$ n ps[[n]] // Simplify

$$\frac{11}{5}-\frac{6^{1-s}}{5}$$

(* autocorrelation function *)

yyb = $\sum_{m=1}^{3}\sum_{n=1}^{3}$ m n p0[[m]] P11[0, s][[m, n]] // Simplify

$$\frac{11}{5}-\frac{6^{1-s}}{5}$$