5.C.2. Random Walk

Random walk was introduced in §4.E-F.

Here, we shall re-write it as a Markov chain.

Consider random walk on a 1-D lattice of spacing Δ & time step τ.

Let

$P(n, s) \equiv P_1(n\Delta, s\tau) \equiv$ Probability of particle being at $x = n\Delta$ at $t = s\tau$.

then (5.20) becomes

$$P(n, s+1) = \sum_{m=-\infty}^{\infty} P(m, s)\, P_{1|1}(m, s \mid n, s+1) \tag{5.41}$$

For a random walker who, at each time step, can go only 1 spacing left or right with equal probability,

$$P_{1|1}(m, s \mid n, s+1) = \frac{1}{2}\delta_{n, m+1} + \frac{1}{2}\delta_{n, m-1} \tag{5.42}$$

(5.41) simplifies to

$$P(n, s+1) = \frac{1}{2} P(n+1, s) + \frac{1}{2} P(n-1, s) \tag{5.43}$$

$$\to \qquad P(n, s+1) - P(n, s) = \frac{1}{2}\left[P(n+1, s) + P(n-1, s) - 2P(n, s) \right]$$

$$\frac{P(n, s+1) - P(n, s)}{\tau} = \frac{\Delta^2}{2\tau}\left[\frac{P(n+1, s) + P(n-1, s) - 2P(n, s)}{\Delta^2} \right] \tag{5.43a}$$

Setting

$$\tau, \ \Delta \to 0 \qquad \text{with} \qquad D = \frac{\Delta^2}{2\tau}$$

(5.43a) becomes

$$\frac{\partial P(x, t)}{\partial t} = D \frac{\partial^2 P(x, t)}{\partial x^2} \tag{5.45}$$

which is just the diffusion equation.

Solution

We shall solve (5.45) for the initial condition

$$P(x, 0) = \delta(x) \tag{5.46a}$$

In terms of the Fourier transform

$$\tilde{P}(k, t) = \int_{-\infty}^{\infty} dx\, e^{ikx} P(x, t) \tag{5.46}$$

(5.45) becomes

$$\frac{\partial \tilde{P}(x, t)}{\partial t} = -D k^2 \tilde{P}(x, t) \tag{5.47}$$

which can be integrated to give

$$\tilde{P}(x, t) = \tilde{P}(x, 0)\, e^{-Dk^2 t} \tag{5.48a}$$

Putting (5.46a) into (5.46) gives

$$\tilde{P}(k, 0) = \int_{-\infty}^{\infty} dx\, e^{ikx} \delta(x) = 1$$

so that (5.48a) becomes

$$\tilde{P}(x,\ t) = e^{-D k^2 t} \tag{5.48}$$

Inverting (5.46) gives

$$\begin{aligned} P(x,\ t) &= \int_{-\infty}^{\infty} \frac{d k}{2\pi} e^{-i k x}\, \tilde{P}(k,\ t) \\ &= \int_{-\infty}^{\infty} \frac{d k}{2\pi} e^{-i k x}\, e^{-D k^2 t} \\ &= \frac{1}{\sqrt{4\pi D t}} e^{-x^2/4 D t} \qquad \text{[see §Code.]} \end{aligned} \tag{5.49}$$

$$\begin{aligned} \langle x(t) \rangle &= \int_{-\infty}^{\infty} d x\, x\, P(x,\ t) \\ &= \frac{1}{\sqrt{4\pi D t}} \int_{-\infty}^{\infty} d x\, x\, e^{-x^2/4 D t} \\ &= 0 \qquad \text{[Integrand is odd in } x. \text{]} \end{aligned}$$

$$\begin{aligned} \langle x^2(t) \rangle &= \int_{-\infty}^{\infty} d x\, x^2\, P(x,\ t) \\ &= \frac{1}{\sqrt{4\pi D t}} \int_{-\infty}^{\infty} d x\, x^2\, e^{-x^2/4 D t} \\ &= 2 D t \qquad \text{[see §Code.]} \end{aligned}$$

Code

In[2]:= $$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i k x}\, e^{-D k^2 t}\, dk$$

Out[2]= ConditionalExpression$\left[\dfrac{e^{-\frac{x^2}{4 D t}}}{2\sqrt{\pi}\ \sqrt{D t}},\ \mathrm{Re}[D t] > 0\right]$

$$\text{Assuming}\left[D \in \text{Reals} \;\&\&\; t \in \text{Reals},\ \frac{1}{\sqrt{4\pi D t}} \int_{-\infty}^{\infty} x^2\, e^{-x^2/(4 D t)}\, dx\right]$$

Out[7]= ConditionalExpression[2 D t, D t > 0]