

5.C.2. Random Walk

Random walk was introduced in §4.E-F.

Here, we shall re-write it as a Markov chain.

Consider random walk on a 1-D lattice of spacing Δ & time step τ .

Let

$$P(n, s) \equiv P_1(n\Delta, s\tau) \equiv \text{Probability of particle being at } x = n\Delta \text{ at } t = s\tau.$$

then (5.20) becomes

$$P(n, s+1) = \sum_{m=-\infty}^{\infty} P(m, s) P_{1|1}(m, s | n, s+1) \quad (5.41)$$

For a random walker who, at each time step, can go only 1 spacing left or right with equal probability,

$$P_{1|1}(m, s | n, s+1) = \frac{1}{2} \delta_{n, m+1} + \frac{1}{2} \delta_{n, m-1} \quad (5.42)$$

(5.41) simplifies to

$$P(n, s+1) = \frac{1}{2} P(n+1, s) + \frac{1}{2} P(n-1, s) \quad (5.43)$$

$$\rightarrow P(n, s+1) - P(n, s) = \frac{1}{2} [P(n+1, s) + P(n-1, s) - 2P(n, s)]$$

$$\frac{P(n, s+1) - P(n, s)}{\tau} = \frac{\Delta^2}{2\tau} \left[\frac{P(n+1, s) + P(n-1, s) - 2P(n, s)}{\Delta^2} \right] \quad (5.43a)$$

Setting

$$\tau, \Delta \rightarrow 0 \quad \text{with} \quad D = \frac{\Delta^2}{2\tau}$$

(5.43a) becomes

$$\frac{\partial P(x, t)}{\partial t} = D \frac{\partial^2 P(x, t)}{\partial x^2} \quad (5.45)$$

which is just the **diffusion equation**.

Solution

We shall solve (5.45) for the initial condition

$$P(x, 0) = \delta(x) \quad (5.46a)$$

In terms of the Fourier transform

$$\tilde{P}(k, t) = \int_{-\infty}^{\infty} dx e^{-ikx} P(x, t) \quad (5.46)$$

(5.45) becomes

$$\frac{\partial \tilde{P}(k, t)}{\partial t} = -Dk^2 \tilde{P}(k, t) \quad (5.47)$$

which can be integrated to give

$$\tilde{P}(k, t) = \tilde{P}(k, 0) e^{-Dk^2 t} \quad (5.48a)$$

Putting (5.46a) into (5.46) gives

$$\tilde{P}(k, 0) = \int_{-\infty}^{\infty} dx e^{-ikx} \delta(x) = 1$$

so that (5.48a) becomes

$$\tilde{P}(k, t) = e^{-Dk^2 t} \quad (5.48)$$

Inverting (5.46) gives

$$\begin{aligned} P(x, t) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tilde{P}(k, t) \\ &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} e^{-Dk^2 t} \\ &= \frac{1}{\sqrt{4\pi Dt}} e^{-x^2/4Dt} \quad [\text{see §Code.}] \quad (5.49) \end{aligned}$$

$$\begin{aligned} \langle x(t) \rangle &= \int_{-\infty}^{\infty} dx x P(x, t) \\ &= \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} dx x e^{-x^2/4Dt} \\ &= 0 \quad [\text{Integrand is odd in } x.] \end{aligned}$$

$$\begin{aligned} \langle x^2(t) \rangle &= \int_{-\infty}^{\infty} dx x^2 P(x, t) \\ &= \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} dx x^2 e^{-x^2/4Dt} \\ &= 2Dt \quad [\text{see §Code.}] \end{aligned}$$

Code

$$\text{In[1]:= } \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} e^{-Dk^2 t} dk$$

$$\text{Out[1]= } \text{ConditionalExpression}\left[\frac{e^{-\frac{x^2}{4Dt}}}{2\sqrt{\pi}\sqrt{Dt}}, \text{Re}[Dt] > 0\right]$$

$$\text{Assuming}[D \in \text{Reals} \ \&\& \ t \in \text{Reals}, \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} x^2 e^{-x^2/(4Dt)} dx]$$

$$\text{ConditionalExpression}[2Dt, Dt > 0]$$