5.C. Markov Chains

A Markov chain is a Markov process at discrete times.

Consider the case where Y can take on M values $\{ y(n),\ n = 1, \ \ldots,\ M \}$, and transitions occur at times $t = s\,\tau$, with $s = 1, \ \ldots, \infty$.

Let

$P(n_1, s_1)$ = Probability that $Y = y(n_1)$ at $t = s_1\,\tau$.

$P_{1|1}(n_1, s_1 \mid n_2, s_2)$ = Conditional probability that

$Y = y(n_2)$ at $t = s_2\,\tau$ given $Y = y(n_1)$ at $t = s_1\,\tau$.

Evolution of the Markov chain is completely determined by P and $P_{1|1}$.

(5.12) becomes

$$P(n, s+1) = \sum_{m=1}^{M} P(m, s)\, P_{1|1}(m, s \mid n, s+1) \tag{5.20}$$

The Chapman-Kolmogorov equation (5.19) takes the form

$$P_{1|1}\left(n_0, s_0 \mid n, s+1\right) = \sum_{m=1}^{M} P_{1|1}(n_0, s_0 \mid m, s)\, P_{1|1}(m, s \mid n, s+1) \tag{5.21}$$

Conditional probability $P_{1|1}(m, s \mid n, s+1)$ is called the transition probability. It controls how the system evolves from one time step to the next, and hence the whole process.

By defining $\boldsymbol{Q}(s \mid s')$ as the matrix with elements

$$Q_{m\,n}(s \mid s') \equiv P_{1|1}(m, s \mid n, s')$$

(5.21) becomes a matrix equation

$$\boldsymbol{Q}(s_0 \mid s+1) = \boldsymbol{Q}(s_0 \mid s)\, \boldsymbol{Q}(s \mid s+1) \tag{5.21a}$$

The transition probabilities then become the transition matrix

$$\boldsymbol{Q}(s) \equiv \boldsymbol{Q}(s \mid s+1)$$

with elements

$$Q_{m\,n}(s) = P_{1|1}(m, s \mid n, s+1) \tag{5.22}$$

In this section, we assume $\boldsymbol{Q}(s)$ to be time independent, i.e., $\boldsymbol{Q}(s) = \boldsymbol{Q}$.

The periodic case, $\boldsymbol{Q}(s+N) = \boldsymbol{Q}(s)$, will be discussed in §S5.A.