5.E.1. Langevin Equation

Consider the Brownian motion in a 1-D fluid. Dynamics of the Brownian particle can be approximately described by the Langevin equations

$$m \frac{d\,v(t)}{d\,t} + \gamma\, v(t) = \xi(t) \tag{5.76}$$

$$\frac{d\,x(t)}{d\,t} = v(t) \tag{5.77}$$

where γ is the coefficient of friction and $\xi(t)$ is a random force due to the density fluctuations in the fluid.

For a Gaussian white noise, $\xi(t)$ is Markovian with

$$\langle\, \xi(t) \,\rangle_\xi = 0 \tag{a}$$

where $\langle\ \rangle_\xi$ is the average with respect to the probability distribution of $\xi(t)$.

A stationary noise is white if it is delta-correlated,

$$\langle\, \xi(t_1)\, \xi(t_2) \,\rangle_\xi = g\, \delta(t_2 - t_1) \tag{5.78}$$

Its power spectrum (Fourier transform) therefore contains all frequency components, and hence white.

According to Exercise 4.9, condition (a) means that all correlation functions with odd number of $\xi(t_j)$ factors vanish while those with even number can be expressed as sums of products of the pairwise correlation function $\langle\, \xi(t_i)\, \xi(t_j) \,\rangle_\xi$. For example,

$$\langle\, \xi(t_1)\, \xi(t_2)\, \xi(t_3)\, \xi(t_4) \,\rangle_\xi = \langle\, \xi(t_1)\, \xi(t_2) \,\rangle_\xi \langle\, \xi(t_3)\, \xi(t_4) \,\rangle_\xi$$
$$+ \langle\, \xi(t_1)\, \xi(t_3) \,\rangle_\xi \langle\, \xi(t_2)\, \xi(t_4) \,\rangle_\xi + \langle\, \xi(t_1)\, \xi(t_4) \,\rangle_\xi \langle\, \xi(t_2)\, \xi(t_3) \,\rangle_\xi$$

(5.77) can be solved using the integrating factor

$$\alpha(t) = e^{(\gamma/m)\,t}$$

so that

$$v(t) = \frac{1}{\alpha(t)} \left(\int_0^t d\,s\, \alpha(s)\, \frac{\xi(s)}{m} + c \right)$$
$$= e^{-(\gamma/m)\,t} \left(\int_0^t d\,s\, e^{(\gamma/m)\,s}\, \frac{\xi(s)}{m} + c \right)$$

Imposing the initial conditions

$$x(0) = x_0 \qquad\qquad v(0) = v_0 \tag{b}$$

we have

$$v(t) = e^{-(\gamma/m)\,t} \left(\int_0^t d\,s\, e^{(\gamma/m)\,s}\, \frac{\xi(s)}{m} + v_0 \right)$$
$$= v_0\, e^{-(\gamma/m)\,t} + \frac{1}{m} \int_0^t d\,s\, e^{-(\gamma/m)(t-s)}\, \xi(s) \tag{5.79}$$

and

$$x(t) = x_0 + \int_0^t d\,s\, v(s)$$
$$= x_0 + \int_0^t d\,s \left[\, v_0\, e^{-(\gamma/m)\,s} + \frac{1}{m} \int_0^s d\,u\, e^{-(\gamma/m)(s-u)}\, \xi(u) \,\right]$$
$$= x_0 + v_0 \frac{m}{\gamma} \left(1 - e^{-(\gamma/m)\,t} \right) + \frac{1}{m} \int_0^t d\,u \int_u^t d\,s\, e^{-(\gamma/m)(s-u)}\, \xi(u)$$

$$= x_0 + v_0 \frac{m}{\gamma}\left(1 - e^{-(\gamma/m)t}\right) + \frac{1}{\gamma}\int_0^t du\left(1 - e^{-(\gamma/m)(t-u)}\right)\xi(u) \qquad (5.80)$$

where the identity

$$\int_0^t ds \int_0^s du = \int_0^t du \int_u^t ds$$

can be easily proved graphically.

Since $\xi(t)$ is a stochastic variable, so are $x(t)$ & $v(t)$.

By (5.79), we have

$$\langle v(t) \rangle_\xi = v_0\, e^{-(\gamma/m)t} + \frac{1}{m}\int_0^t ds\, e^{-(\gamma/m)(t-s)} \langle \xi(s) \rangle_\xi$$

$$= v_0\, e^{-(\gamma/m)t} \qquad \text{[(a) used.]}$$

$\rightarrow \qquad \langle v(0) \rangle_\xi = v_0$

in agreement with (b).

Similarly, (5.80) gives

$$\langle x(t) \rangle_\xi = x_0 + v_0 \frac{m}{\gamma}\left(1 - e^{-(\gamma/m)t}\right) + \frac{1}{\gamma}\int_0^t du\left(1 - e^{-(\gamma/m)(t-u)}\right)\langle \xi(u) \rangle_\xi$$

$$= x_0 + v_0 \frac{m}{\gamma}\left(1 - e^{-(\gamma/m)t}\right)$$

$\rightarrow \qquad \langle [x(t) - x_0] \rangle_\xi = v_0 \dfrac{m}{\gamma}\left(1 - e^{-(\gamma/m)t}\right)$

Using (a), (5.78) and

$$v(t_2)\, v(t_1) = v_0^2\, e^{-(\gamma/m)(t_2+t_1)} + v_0\, e^{-(\gamma/m)t_2} \frac{1}{m}\int_0^{t_1} ds_1\, e^{-(\gamma/m)(t_1-s_1)}\, \xi(s_1)$$

$$+ v_0\, e^{-(\gamma/m)t_1} \frac{1}{m}\int_0^{t_2} ds_2\, e^{-(\gamma/m)(t_2-s_2)}\, \xi(s_2)$$

$$+ \frac{1}{m^2}\int_0^{t_2} ds_2 \int_0^{t_1} ds_1\, e^{-(\gamma/m)(t_2-s_2)}\, e^{-(\gamma/m)(t_1-s_1)}\, \xi(s_1)\, \xi(s_2)$$

we have

$$\langle v(t_2)\, v(t_1) \rangle_\xi = v_0^2\, e^{-(\gamma/m)(t_2+t_1)}$$

$$+ \frac{1}{m^2} e^{-(\gamma/m)(t_2+t_1)} \int_0^{t_2} ds_2 \int_0^{t_1} ds_1\, e^{(\gamma/m)(s_1+s_2)}\, g\, \delta(s_2 - s_1)$$

Since the condition $s_2 = s_1$ can be satisfied only up to $s_1 = s_2 = \min(t_1, t_2) \equiv T$, we have

$$\langle v(t_2)\, v(t_1) \rangle_\xi = v_0^2\, e^{-(\gamma/m)(t_2+t_1)} + \frac{g}{m^2} e^{-(\gamma/m)(t_2+t_1)} \int_0^T ds\, e^{2(\gamma/m)s}$$

$$= v_0^2\, e^{-(\gamma/m)(t_2+t_1)} + \frac{g}{2m\gamma} e^{-(\gamma/m)(t_2+t_1)}\left(e^{2(\gamma/m)T} - 1\right)$$

$$= \left(v_0^2 - \frac{g}{2m\gamma}\right) e^{-(\gamma/m)(t_2+t_1)} + \frac{g}{2m\gamma} e^{-(\gamma/m)|t_2-t_1|} \qquad (5.82)$$

Similarly,

$$\langle\, [x(t_2) - x_0]\ [x(t_1) - x_0]\, \rangle_\xi$$

$$= v_0^2 \frac{m^2}{\gamma^2} \left(1 - e^{-(\gamma/m) t_2}\right)\left(1 - e^{-(\gamma/m) t_1}\right)$$

$$+ \frac{1}{\gamma^2} \int_0^{t_2} d u_2 \int_0^{t_1} d u_1 \left(1 - e^{-(\gamma/m)(t_2 - u_2)}\right)\left(1 - e^{-(\gamma/m)(t_1 - u_1)}\right) g\, \delta(u_2 - u_1)$$

Working on the 2nd term gives

$$\mathcal{I} = \frac{g}{\gamma^2} \int_0^T d u \left(1 - e^{-(\gamma/m)(t_2 - u)}\right)\left(1 - e^{-(\gamma/m)(t_1 - u)}\right)$$

$$= \frac{g}{\gamma^2} \int_0^T d u \left(1 - e^{-(\gamma/m)(t_2 - u)} - e^{-(\gamma/m)(t_1 - u)} + e^{-(\gamma/m)(t_2 + t_1)} e^{2(\gamma/m) u} \right)$$

$$= \frac{g}{\gamma^2} \left[T - \frac{m}{\gamma} e^{-(\gamma/m) t_2} \left(e^{(\gamma/m) T} - 1\right) - \frac{m}{\gamma} e^{-(\gamma/m) t_1} \left(e^{(\gamma/m) T} - 1\right) \right.$$

$$\left. + \frac{m}{2\gamma} e^{-(\gamma/m)(t_2 + t_1)} \left(e^{2(\gamma/m) T} - 1\right) \right]$$

$$= \frac{g}{\gamma^2} T - \frac{g m}{\gamma^3} \left(e^{-(\gamma/m) t_2} + e^{-(\gamma/m) t_1}\right)\left(e^{(\gamma/m) T} - 1\right)$$

$$+ \frac{g m}{2 \gamma^3} e^{-(\gamma/m)(t_2 + t_1)} \left(e^{2(\gamma/m) T} - 1\right)$$

For $t_2 > t_1$, we have

$$\mathcal{I} = \frac{g}{\gamma^2} t_1 - \frac{g m}{\gamma^3} \left(e^{-(\gamma/m)(t_2 - t_1)} - e^{-(\gamma/m) t_2} - e^{-(\gamma/m) t_1} + 1\right)$$

$$+ \frac{g m}{2 \gamma^3} \left(e^{-(\gamma/m)(t_2 - t_1)} - e^{-(\gamma/m)(t_2 + t_1)} \right)$$

$$= \frac{g}{\gamma^2} t_1 - \frac{g m}{\gamma^3} \left(\frac{1}{2} e^{-(\gamma/m)(t_2 - t_1)} + \frac{1}{2} e^{-(\gamma/m)(t_2 + t_1)} - e^{-(\gamma/m) t_2} - e^{-(\gamma/m) t_1} + 1\right)$$

Since the $t_1 < t_2$ case can be obtained by the substitution $t_2 \leftrightarrow t_1$, the general case is simply

$$\mathcal{I} = \frac{g}{\gamma^2} T - \frac{g m}{\gamma^3} \left(\frac{1}{2} e^{-(\gamma/m)|t_2 - t_1|} + \frac{1}{2} e^{-(\gamma/m)(t_2 + t_1)} - e^{-(\gamma/m) t_2} - e^{-(\gamma/m) t_1} + 1\right)$$

Therefore

$$\langle\, [x(t_2) - x_0]\, [x(t_1) - x_0]\, \rangle_\xi$$

$$= v_0^2 \frac{m^2}{\gamma^2} \left(1 - e^{-(\gamma/m) t_2}\right)\left(1 - e^{-(\gamma/m) t_1}\right)$$

$$+ \frac{g}{\gamma^2} T - \frac{g m}{\gamma^3} \left(\frac{1}{2} e^{-(\gamma/m)|t_2 - t_1|} + \frac{1}{2} e^{-(\gamma/m)(t_2 + t_1)} - e^{-(\gamma/m) t_2} - e^{-(\gamma/m) t_1} + 1\right)$$

Setting $t_2 = t_1 = t$ gives

$$\left\langle\, [x(t) - x_0]^2 \,\right\rangle_\xi$$

$$= v_0^2 \frac{m^2}{\gamma^2} \left(1 - e^{-(\gamma/m) t}\right)^2 + \frac{g}{\gamma^2} t - \frac{g m}{\gamma^3} \left(\frac{1}{2} + \frac{1}{2} e^{-2(\gamma/m) t} - 2 e^{-(\gamma/m) t} + 1\right)$$

$$= v_0^2 \frac{m^2}{\gamma^2} \left(1 - e^{-(\gamma/m) t}\right)^2 + \frac{g}{\gamma^2} t - \frac{g m}{\gamma^3} \left(\frac{1}{2} \left(1 - e^{-(\gamma/m) t}\right)^2 - e^{-(\gamma/m) t} + 1\right)$$

$$= \frac{m^2}{\gamma^2} \left(v_0^2 - \frac{g}{2 m \gamma} \right)\left(1 - e^{-(\gamma/m) t}\right)^2 + \frac{g}{\gamma^2} \left[t - \frac{m}{\gamma} \left(1 - e^{-(\gamma/m) t}\right)\right] \qquad (5.83)$$

As $t \to \infty$, we have

$$\left\langle\, [x(t) - x_0]^2 \,\right\rangle_\xi \to \frac{m^2}{\gamma^2} \left(v_0^2 - \frac{3 g}{2 m \gamma} \right) + \frac{g}{\gamma^2} t$$

$$\to \frac{g}{\gamma^2} t$$

which agrees with the random walk result if we set $D = \frac{g}{2\gamma^2}$.

If the Brownian particle is in equilibrium with the fluid, $v(t)$ must be a stationary distribution. Therefore, $\langle v(t_2)\, v(t_1) \rangle_\xi$ must be a function of $|\, t_2 - t_1 \,|$. The 1st term in (5.82) thus vanishes, giving

$$v_0^2 - \frac{g}{2 m \gamma} = 0$$

Using the equipartition theorem, we have

$$\frac{1}{2} m \langle v_0^2 \rangle_T = \frac{1}{2} k_B T$$

so that

$$\langle g \rangle_T = 2 m \gamma \langle v_0^2 \rangle_T = 2 \gamma k_B T \qquad \text{(b)}$$

(5.82) then becomes

$$\langle \langle v(t_2)\, v(t_1) \rangle_\xi \rangle_T = \frac{k_B T}{m} e^{-(\gamma/m)|t_2 - t_1|} \qquad (5.84)$$

By the same token, (5.83) becomes

$$\langle \langle [x(t) - x_0]^2 \rangle_\xi \rangle_T = \frac{2 k_B T}{\gamma} \left[t - \frac{m}{\gamma} \left(1 - e^{-(\gamma/m) t} \right) \right] \qquad (5.85)$$

As $t \to \infty$, we have

$$\langle \langle [x(t) - x_0]^2 \rangle_\xi \rangle_T \to \frac{2 k_B T}{\gamma} t$$

(b) also gives

$$\langle D \rangle_T = \frac{\langle g \rangle_T}{2 \gamma^2} = \frac{k_B T}{\gamma}$$

For spherical Brownian particles, we have

$$\gamma = 6 \pi \eta a$$

where η is the shear viscosity of the fluid and a the radius of each particle. We then have

$$\langle D \rangle_T = \frac{k_B T}{6 \pi \eta a} = \frac{R T}{6 \pi \eta a N_A} \qquad \text{(c)}$$

where R is the gas constant and N_A the Avogadro's number. (c) was first derived by Einstein.

Exercise 5.5

Consider a Brownian particle of mass m which is attached to a harmonic spring with force constant k and is constrained to move in 1-D. The Langevin equations are

$$m \frac{d v(t)}{d t} + \gamma v(t) + m \omega_0^2 x(t) = \xi(t) \qquad \text{and} \qquad \frac{d x(t)}{d t} = v(t)$$

where $\omega_0 = \sqrt{\frac{k}{m}}$. Let the initial position and velocity of the Brownian particle be x_0 and v_0, respectively, and assume that it is initially in equilibrium with the fluid. By the equipartition theorem, the average kinetic energy is $\frac{1}{2} m \langle v_0^2 \rangle_T = \frac{1}{2} k_B T$ and the average vibrational energy,

$\frac{1}{2} m\,\omega_0^2 \langle x_0^2 \rangle_T = \frac{1}{2} k_B T$. We also assume x_0 and v_0 to be statistically independent so that $\langle x_0 v_0 \rangle_T = 0$.

(a) Show that a condition for the process to be stationary is that the noise strength is $g = 4\,\gamma\,k_B T$.

(b) Compute the velocity correlation function $\langle\langle v(t_2)\, v(t_1) \rangle_\xi\rangle_T$.

Answer

The Langevin equation can be solved using Laplace transforms defined by

$$V(s) = \int_0^\infty d t\, e^{-s t} v(t) \qquad\qquad \Xi(s) = \int_0^\infty d t\, e^{-s t} \xi(t)$$

Using

$$\int_0^\infty d t\, e^{-s t} \frac{d\, v(t)}{d t} = s\, V(s) - v_0$$

$$\int_0^\infty d t\, e^{-s t} x(t) = \int_0^\infty d t\, e^{-s t} \left(x_0 + \int_0^t d t'\, v(t') \right) = \frac{1}{s} x_0 + \frac{1}{s} V(s)$$

we have

$$m \left(s\, V(s) - v_0 \right) + \gamma\, V(s) + m\,\omega_0^2 \left(x_0 + \frac{1}{s} V(s) \right) = \Xi(s)$$

$$\rightarrow \qquad V(s) = \frac{s}{s^2 + \Gamma s + \omega_0^2} \left(v_0 - \frac{1}{s} \omega_0^2\, x_0 + \frac{1}{m} \Xi(s) \right) \qquad\qquad \Gamma = \frac{\gamma}{m}$$

Taking the inverse transform, we have [see Arfken]

$$v(t) = \frac{1}{2 \pi i} \int_{\delta - i\infty}^{\delta + i\infty} d s\, e^{s t}\, V(s)$$

where δ is chosen to put all singularities of $V(s)$ to the left of the integration path. Since we close the integration contour with a great half-circle of infinite radius on the left hand side of the complex plane, this means all singularities of $V(s)$ are included in the residue sum.

Using

$$s^2 + \Gamma s + \omega_0^2 = (s + \Gamma + \Delta)(s + \Gamma - \Delta)$$

where

$$\Delta \equiv \sqrt{\Gamma^2 - \omega_0^2} = \sqrt{\frac{\gamma^2}{4 m^2} - \omega_0^2}$$

we have

$$\frac{1}{2 \pi i} \int_{\delta - i\infty}^{\delta + i\infty} d s\, e^{s t} \frac{1}{s^2 + \Gamma s + \omega_0^2}$$

$$= e^{-(\Gamma + \Delta) t} \frac{1}{2 \Delta} - e^{-(\Gamma - \Delta) t} \frac{1}{2 \Delta}$$

$$= -\frac{1}{\Delta} e^{-\Gamma t} \sinh \Delta t$$

$$\frac{1}{2 \pi i} \int_{\delta - i\infty}^{\delta + i\infty} d s\, e^{s t} \frac{s}{s^2 + \Gamma s + \omega_0^2}$$

$$= e^{-(\Gamma + \Delta) t} \frac{\Gamma + \Delta}{2 \Delta} - e^{-(\Gamma - \Delta) t} \frac{\Gamma - \Delta}{2 \Delta}$$

$$= e^{-\Gamma t} \left[\frac{\Gamma}{\Delta} \left(\frac{e^{-\Delta t} - e^{\Delta t}}{2} \right) + \frac{e^{-\Delta t} + e^{\Delta t}}{2} \right]$$

$$= e^{-\Gamma t}\left(-\frac{\Gamma}{\Delta}\sinh\Delta t + \cosh\Delta t\right)$$
$$= e^{-\Gamma t} C(t)$$

where

$$C(t) \equiv \cosh\Delta t - \frac{\Gamma}{\Delta}\sinh\Delta t$$

and

$$\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} ds\, e^{st}\frac{s}{s^2+\Gamma s+\omega_0^2}\,\Xi(s)$$
$$= \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} ds\, e^{st}\frac{s}{s^2+\Gamma s+\omega_0^2}\int_0^\infty dt'\, e^{-st'}\,\xi(t')$$
$$= \frac{1}{2\pi i}\int_0^\infty dt'\,\xi(t')\int_{\delta-i\infty}^{\delta+i\infty} ds\, e^{s(t-t')}\frac{s}{s^2+\Gamma s+\omega_0^2}$$
$$= \int_0^t dt'\,\xi(t')\, e^{-\Gamma(t-t')}\, C(t-t')$$

where the upper integration limit of t' is changed since for $t'>t$, the factor $e^{s(t-t')}$ requires the half-circle part of the contour to lie on the right hand side so that $\mathrm{Re}\, s>0$. This means the contour encloses no poles and the integral vanishes.

Hence,

$$v(t) = v_0\, e^{-\Gamma t} C(t) - \frac{1}{\Delta}\omega_0^2 x_0\, e^{-\Gamma t}\sinh\Delta t + \frac{1}{m}\int_0^t dt'\,\xi(t')\, e^{-\Gamma(t-t')}\, C(t-t')$$

Using

$$\langle \xi(t)\rangle_\xi = 0 \qquad \langle \xi(t)\,\xi(t')\rangle_\xi = g\,\delta(t-t')$$

we have

$$\langle v(t_2)\, v(t_1)\rangle_\xi$$
$$= \left[v_0\, e^{-\Gamma t_2} C(t_2) - \frac{1}{\Delta}\omega_0^2 x_0\, e^{-\Gamma t_2}\sinh\Delta t_2\right]\left[v_0\, e^{-\Gamma t_1} C(t_1) - \frac{1}{\Delta}\omega_0^2 x_0\, e^{-\Gamma t_1}\sinh\Delta t_1\right]$$
$$+\frac{1}{m^2} g\int_0^{t_2} dt'\int_0^{t_1} dt''\, e^{-\Gamma(t_2-t')}\, e^{-\Gamma(t_1-t'')}\, C(t_2-t')\, C(t_1-t'')\,\delta(t'-t'')$$
$$= \left[v_0\, e^{-\Gamma t_2} C(t_2) - \frac{1}{\Delta}\omega_0^2 x_0\, e^{-\Gamma t_2}\sinh\Delta t_2\right]\left[v_0\, e^{-\Gamma t_1} C(t_1) - \frac{1}{\Delta}\omega_0^2 x_0\, e^{-\Gamma t_1}\sinh\Delta t_1\right]$$
$$+\frac{1}{m^2} g\int_0^{T} dt\, e^{-\Gamma(t_2-t)}\, e^{-\Gamma(t_1-t)}\, C(t_2-t)\, C(t_1-t)$$

where

$$T = \min(t_2, t_1)$$

Using

$$\langle v_0^2\rangle_T = \frac{k_B T}{m} \qquad \langle x_0^2\rangle_T = \frac{k_B T}{m\,\omega_0^2} \quad \langle x_0\, v_0\rangle_T = 0$$
$$\omega_0^2 = \Gamma^2 - \Delta^2$$

we have

$$\left\langle \langle v(t_2)\, v(t_1) \rangle_\xi \right\rangle_T = \frac{k_B T}{m} e^{-\Gamma(t_2+t_1)} C(t_2)\, C(t_1)$$
$$+\frac{1}{\Delta^2}\frac{k_B T}{m}\left(\Gamma^2-\Delta^2\right) e^{-\Gamma(t_2+t_1)} \sinh\Delta\, t_2 \, \sinh\Delta\, t_1$$
$$+\frac{1}{m^2} g \int_0^T d\,t\; e^{-\Gamma(t_2+t_1-2t)} C(t_2-t)\, C(t_1-t)$$

The integral can be evaluated using *Mathematica*. Assuming $t_2 > t_1$, we have [see §Code] :

$$\mathcal{I} = \frac{1}{4\Gamma\Delta^2} e^{-(t_1+t_2)\Gamma}\Big\{ \left[\Gamma^2 + \left(-1+e^{2t_1\Gamma}\right)\Delta^2\right]\cosh\Delta(t_1-t_2)$$
$$-\Gamma^2\cosh\Delta(t_1+t_2) + \Gamma\Delta\left(e^{2t_1\Gamma}\sinh\Delta(t_1-t_2) + \sinh\Delta(t_1+t_2)\right)\Big\}$$
$$= \frac{1}{4\Gamma\Delta^2} e^{-(t_1+t_2)\Gamma}\Big\{ \left(\Gamma^2-\Delta^2\right)\cosh\Delta(t_2-t_1)$$
$$-\Gamma^2\cosh\Delta(t_2+t_1) + \Gamma\Delta\sinh\Delta(t_2+t_1)\Big\}$$
$$+\frac{1}{4} e^{-(t_2-t_1)\Gamma}\left\{\frac{1}{\Gamma}\cosh\Delta(t_2-t_1) - \frac{1}{\Delta}\sinh\Delta(t_2-t_1)\right\}$$

Using

$$C(t_2)\, C(t_1) = \frac{1}{2\Delta^2}\Big\{ \left(\Delta^2-\Gamma^2\right)\cosh\Delta(t_2-t_1) + \left(\Delta^2+\Gamma^2\right)\cosh\Delta(t_2+t_1)$$
$$-2\Gamma\Delta\sinh\Delta(t_2+t_1)\Big\}$$
$$\sinh\Delta\, t_2\, \sinh\Delta\, t_1 = \frac{1}{2}\left\{\cosh\Delta(t_2+t_1) - \cosh\Delta(t_2-t_1)\right\}$$

the terms in $\left\langle \langle v(t_2)\, v(t_1) \rangle_\xi \right\rangle_T$ that depend on t_2+t_1 become

$$\mathcal{J} = \frac{k_B T}{m} e^{-\Gamma(t_2+t_1)}\Big\{ C(t_2)\, C(t_1) + \frac{1}{\Delta^2}\left(\Gamma^2-\Delta^2\right)\sinh\Delta\, t_2\, \sinh\Delta\, t_1$$
$$+\frac{g}{4\, m\, k_B T\,\Gamma\Delta^2}\Big[\left(\Gamma^2-\Delta^2\right)\cosh\Delta(t_2-t_1)$$
$$-\Gamma^2\cosh\Delta(t_2+t_1) + \Gamma\Delta\sinh\Delta(t_2+t_1)\Big]\Big\}$$
$$= \frac{e^{-\Gamma(t_2+t_1)}}{2\Delta^2}\frac{k_B T}{m}\Big\{$$
$$\left(\Delta^2-\Gamma^2\right)\cosh\Delta(t_2-t_1) + \left(\Delta^2+\Gamma^2\right)\cosh\Delta(t_2+t_1) - 2\Gamma\Delta\sinh\Delta(t_2+t_1)$$
$$+\left(\Gamma^2-\Delta^2\right)\left[\cosh\Delta(t_2+t_1) - \cosh\Delta(t_2-t_1)\right]$$
$$+\frac{g}{2\, m\, k_B T\,\Gamma}\left[\left(\Gamma^2-\Delta^2\right)\cosh\Delta(t_2-t_1) - \Gamma^2\cosh\Delta(t_2+t_1) + \Gamma\Delta\sinh\Delta(t_2+t_1)\right]\Big\}$$
$$= \frac{e^{-\Gamma(t_2+t_1)}}{\Delta^2}\frac{k_B T}{2m}\Big\{ \left(\Delta^2-\Gamma^2\right)\left(2-\frac{g}{2\Gamma\, m\, k_B T}\right)\cosh\Delta(t_2-t_1)$$
$$+\left(2\Gamma^2 - \frac{g\Gamma}{2\, m\, k_B T}\right)\cosh\Delta(t_2+t_1)$$
$$+\left(-2\Gamma\Delta + \frac{g\Delta}{2\, m\, k_B T}\right)\sinh\Delta(t_2+t_1)\Big\}$$

$$= \frac{e^{-\Gamma(t_2+t_1)}}{\Delta^2} \frac{k_B T}{2m} \left(2 - \frac{g}{2\Gamma m k_B T}\right)$$
$$\times\left\{ (\Delta^2 - \Gamma^2) \cosh \Delta(t_2 - t_1) + \Gamma^2 \cosh \Delta(t_2 + t_1) - \Gamma \Delta \sinh \Delta(t_2 + t_1) \right\}$$

Setting

$$g = 4\,\Gamma\, m\, k_B T = 4\,\gamma\, k_B T$$

thus makes $\mathcal{J} = 0$ so that

$$\langle\langle v(t_2)\, v(t_1)\rangle_\xi\rangle_T = \frac{g}{4m^2} e^{-(t_2-t_1)\Gamma}\left\{ \frac{1}{\Gamma} \cosh \Delta(t_2 - t_1) - \frac{1}{\Delta} \sinh \Delta(t_2 - t_1) \right\}$$
$$= \frac{k_B T}{m} e^{-(t_2-t_1)\Gamma}\left\{ \cosh \Delta(t_2 - t_1) - \frac{\Gamma}{\Delta} \sinh \Delta(t_2 - t_1) \right\}$$

is stationary.

The case $t_1 > t_2$ is obtained by the substitution $t_1 \leftrightarrow t_2$, giving

$$\langle\langle v(t_2)\, v(t_1)\rangle_\xi\rangle_T = \frac{k_B T}{m} e^{-(t_1-t_2)\Gamma}\left\{ \frac{1}{\Gamma} \cosh \Delta(t_1 - t_2) - \frac{1}{\Delta} \sinh \Delta(t_1 - t_2) \right\}$$

Combining both cases gives

$$\langle\langle v(t_2)\, v(t_1)\rangle_\xi\rangle_T = \frac{k_B T}{m} e^{-|t_2-t_1|\Gamma}\left\{ \cosh \Delta \left| t_2 - t_1 \right| - \frac{\Gamma}{\Delta} \sinh \Delta \left| t_2 - t_1 \right| \right\}$$
$$\rightarrow \quad \langle\langle v(t_1+\tau)\, v(t_1)\rangle_\xi\rangle_T = \frac{k_B T}{m} e^{-|\tau|\Gamma}\left\{ \cosh \Delta \left| \tau \right| - \frac{\Gamma}{\Delta} \sinh \Delta \left| \tau \right| \right\}$$

Code

In[1]:=
```
CC[t_] := Cosh[Δ t] - Γ/Δ Sinh[Δ t]
```

In[2]:=
```
𝓘 = ∫_0^t1 ⅇ^(-Γ (t2+t1-2 t)) CC[t2 - t] CC[t1 - t] ⅆt
```

Out[2]=
$$\frac{1}{4\Gamma\Delta^2} e^{-(t1+t2)\Gamma} \left(\left(\Gamma^2 + \left(-1 + e^{2\,t1\,\Gamma}\right)\Delta^2\right) \mathrm{Cosh}\left[(t1 - t2)\,\Delta\right] + \Gamma\left(-\Gamma\, \mathrm{Cosh}\left[(t1 + t2)\,\Delta\right] + \Delta\left(e^{2\,t1\,\Gamma}\, \mathrm{Sinh}\left[(t1 - t2)\,\Delta\right] + \mathrm{Sinh}\left[(t1 + t2)\,\Delta\right]\right)\right)\right)$$

In[15]:=
```
CC[t₂] CC[t₁] // Expand // TrigReduce
```

Out[15]=
$$\frac{1}{2\Delta^2}\left(-\Gamma^2 \mathrm{Cosh}[\Delta t_1 - \Delta t_2] + \Delta^2 \mathrm{Cosh}[\Delta t_1 - \Delta t_2] + \Gamma^2 \mathrm{Cosh}[\Delta t_1 + \Delta t_2] + \Delta^2 \mathrm{Cosh}[\Delta t_1 + \Delta t_2] - 2\,\Gamma\,\Delta\, \mathrm{Sinh}[\Delta t_1 + \Delta t_2]\right)$$

In[4]:=
```
Sinh[Δ t₂] Sinh[Δ t₁] // TrigReduce
```

Out[4]=
$$\frac{1}{2}\left(-\mathrm{Cosh}[\Delta t_1 - \Delta t_2] + \mathrm{Cosh}[\Delta t_1 + \Delta t_2]\right)$$

In[9]:= $\frac{k\,T}{m}\, e^{-\Gamma\,(t2+t1)}$ **CC[t2] CC[t1]** + $\frac{k\,T}{m\,\Delta^2}\,\left(\Gamma^2-\Delta^2\right)\, e^{-\Gamma\,(t2+t1)}$ **Sinh[Δ t2] Sinh[Δ t1]** + $\frac{g}{m^2}\,\mathcal{I}$ **//**
Expand // Together // Simplify

Out[9]= $\frac{1}{4\,m^2\,\Gamma\,\Delta^2} e^{-(t1+t2)\,\Gamma}$
$\Big(\big(4\,k\,m\,T\,\Gamma\,\left(-\Gamma^2+\Delta^2\right)+g\,\left(\Gamma^2+\left(-1+e^{2\,t1\,\Gamma}\right)\,\Delta^2\right)\big)\,\mathrm{Cosh}\big[(t1-t2)\,\Delta\big]+\Gamma\,\left(\Gamma\,\left(-g+4\,k\,m\,T\,\Gamma\right)\right.$
$\left.\mathrm{Cosh}\big[(t1+t2)\,\Delta\big]+\Delta\,\left(e^{2\,t1\,\Gamma}\,g\,\mathrm{Sinh}\big[(t1-t2)\,\Delta\big]+\left(g-4\,k\,m\,T\,\Gamma\right)\,\mathrm{Sinh}\big[(t1+t2)\,\Delta\big]\right)\right)\Big)$