7.F. Order-Disorder Transitions

See Chap 3, especially §3.G, for an introduction to continuous phase transitions.

Consider a lattice of N spin-$\frac{1}{2}$ atoms with magnetic moment μ. In the Ising model, each spin can only be parallel ($s = 1$) or anti-parallel ($s = -1$) to the direction of a magnetic field $\boldsymbol{B}$. Furthermore, only nearest neighbor interactions are allowed so that

$$H = \sum_{\{i,j\}} \epsilon_{ij}\, s_i\, s_j - \mu B \sum_{i=1}^{N} s_i \qquad s_i = \pm 1 \qquad (7.75)$$

where ϵ_{ij} is the interaction energy and $\sum_{\{i,j\}}$ denotes the sum over nearest-neighbors. For 1 & 2-D lattices, there is always a quantization direction along which we can set ϵ_{ij} to be the same for all i, j. In a 3-D lattice, in-plane and off-plane interactions for spins on any plane are necessarily different for topological reasons. However, this difference is often ignored for the sake of simplicity.

Note that for $B = 0$, the ground state is

ferromagnetic (all spins are in the same direction) if $\epsilon_{ij} < 0$,

and

antiferromagnetic (all neighboring spins are in opposite directions) if $\epsilon_{ij} > 0$.

The partition function is

$$Z_N(T) = \sum_{\text{all spin states}} e^{-\beta H}$$

$$= \sum_{s_1=\pm1} \cdots \sum_{s_N=\pm1} \exp\left[-\beta \left(\sum_{\{i,j\}} \epsilon_{ij}\, s_i\, s_j - \mu B \sum_{i=1}^{N} s_i \right) \right] \qquad (7.76)$$

Read Reichl's text.

7.F.1. Exact Solution for a 1-D Lattice

Consider the ferromagnetic case. To avoid dealing with the special situation at the end points, we impose the periodic boundary conditions,

$$s_{i+N} = s_i \qquad \forall\ i \qquad (7.77a)$$

so that (7.75) takes the simple form

$$H = -\epsilon \sum_{i=1}^{N} s_i\, s_{i+1} - \mu B \sum_{i=1}^{N} s_i \qquad \epsilon > 0 \qquad (7.77)$$

$$= -\sum_{i=1}^{N} \left[\epsilon\, s_i\, s_{i+1} + \frac{1}{2} \mu B \,(s_i + s_{i+1}) \right] \qquad (7.77b)$$

Note: One way to achieve the periodic boundary conditions is to join the ends of the 1-D lattice to form a ring. The same trick can be applied to each axis of a periodic lattice of arbitrary dimensions, though the result may not be so easy to visualize.

Consider now

$$H_{i,i+1} = -\epsilon\, s_i\, s_{i+1} - \frac{1}{2} \mu B \,(s_i + s_{i+1})$$

Its values are tabulated as follows

$H_{i,i+1}$	$s_{i+1}=1$	$s_{i+1}=-1$
$s_i=1$	$-\epsilon-\mu B$	ϵ
$s_i=-1$	ϵ	$-\epsilon+\mu B$

(7.77c)

which means $H_{i,i+1}$ can be represented by the matrix

$$\mathbb{T}_{i,i+1} = -\begin{pmatrix} \epsilon+\mu B & -\epsilon \\ -\epsilon & \epsilon-\mu B \end{pmatrix} \tag{7.79a}$$

with

$$(\,1 \quad 0\,) = \left\langle s_i = 1 \,\right| \qquad (\,0 \quad 1\,) = \left\langle s_i = -1 \,\right|$$
$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left|\, s_{i+1} = 1 \right\rangle \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \left|\, s_{i+1} = -1 \right\rangle \tag{7.79b}$$

Thus, $\mathbb{T}_{i,i+1}$ is simply the matrix representation of the quantum operator $\hat{T}_{i,i+1}$ whose matrix elements are

$$\begin{aligned}\left\langle s_m \mid \hat{T}_{i,i+1} \mid s_n \right\rangle &= \delta_{m\,i}\,\delta_{n,i+1}\left\langle s_i \mid \hat{T}_{i,i+1} \mid s_{i+1}\right\rangle \\ &= \delta_{m\,i}\,\delta_{n,i+1}\,H_{i,i+1} \\ &= \delta_{m\,i}\,\delta_{n,i+1}\left[-\epsilon\, s_i\, s_{i+1} - \frac{1}{2}\mu B\,(s_i + s_{i+1})\right]\end{aligned} \tag{7.80a}$$

$\hat{T}_{i,i+1}$ is sometimes called the transfer operator linking site i to site $i+1$.

Caution: $\hat{T}_{i,i+1}$ cannot be obtained from $H_{i,i+1}$ by replacing s_i with $\hat{s}_i$. For example,

$$\langle s_i \mid s_{i+1}\rangle = 0 \quad\rightarrow\quad \left\langle s_i \,\middle|\, \hat{s}_i + \hat{s}_{i+1} \,\middle|\, s_{i+1}\right\rangle = (s_i + s_{i+1})\langle s_i \mid s_{i+1}\rangle = 0$$

is at odds with (7.80a).

For the 1-D version of the partition function (7.76),

$$\begin{aligned} Z_N(T,B) &= \sum_{s_1=\pm1} \cdots \sum_{s_N=\pm1} e^{-\beta H} \\ &= \sum_{s_1=\pm1} \cdots \sum_{s_N=\pm1} \exp\left[-\beta \sum_{i=1}^{N}\left(\epsilon\, s_i\, s_{i+1} - \frac{1}{2}\mu B\,(s_i + s_{i+1})\right)\right]\end{aligned} \tag{7.78}$$

we can likewise define a transfer matrix

$$\mathbb{P}_{i,i+1} = \begin{pmatrix} e^{\beta(\epsilon+\mu B)} & e^{-\beta\epsilon} \\ e^{-\beta\epsilon} & e^{\beta(\epsilon-\mu B)} \end{pmatrix} = \mathbb{P} \tag{7.79}$$

and the corresponding transfer operator $\hat{P}_{i,i+1}$ with matrix elements

$$\begin{aligned}\left\langle s_m \mid \hat{P}_{i,i+1} \mid s_n \right\rangle &= \delta_{m\,i}\,\delta_{n,i+1}\left\langle s_i \mid \hat{P}_{i,i+1} \mid s_{i+1}\right\rangle \\ &= \delta_{m\,i}\,\delta_{n,i+1}\,e^{-\beta H_{i,i+1}} \\ &= \delta_{m\,i}\,\delta_{n,i+1}\exp\left\{\beta\left[\epsilon\, s_i\, s_{i+1} + \frac{1}{2}\mu B\,(s_i + s_{i+1})\right]\right\}\end{aligned} \tag{7.80}$$

Putting (7.80) into (7.78) gives

$$\begin{aligned} Z_N(T,B) &= \sum_{s_1=\pm1} \cdots \sum_{s_N=\pm1} \left\langle s_1 \mid \hat{P}_{1,2} \mid s_2\right\rangle\left\langle s_2 \mid \hat{P}_{2,3} \mid s_3\right\rangle \ldots \left\langle s_N \mid \hat{P}_{N,1} \mid s_1\right\rangle \\ &= \mathrm{Tr}\left(\mathbb{P}_{1,2}\,\mathbb{P}_{2,3}\ldots\mathbb{P}_{N,1}\right) && [\,(7.80b)\text{ used.}\,] \\ &= \mathrm{Tr}\,\mathbb{P}^N && [\,(7.79)\text{ used.}\,] \\ &= \lambda_+^N + \lambda_-^N \end{aligned} \tag{7.81a}$$

where $\lambda_\pm$ are the eigenvalues of $\mathbb{P}$. From (7.79), we get the secular equation

$$\left[e^{\beta(\epsilon+\mu B)} - \lambda\right]\left[e^{\beta(\epsilon-\mu B)} - \lambda\right] - e^{-2\beta\epsilon} = 0$$
$$\rightarrow \quad \lambda^2 - \left[e^{\beta(\epsilon+\mu B)} + e^{\beta(\epsilon-\mu B)}\right]\lambda + e^{2\beta\epsilon} - e^{-2\beta\epsilon} = 0$$
$$\lambda^2 - 2\cosh(\beta\mu B)\,e^{\beta\epsilon}\,\lambda + 2\sinh(2\beta\epsilon) = 0$$

$$\therefore \quad \lambda_\pm = \cosh(\beta\mu B)\, e^{\beta\epsilon} \pm \sqrt{\cosh^2(\beta\mu B)\, e^{2\beta\epsilon} - 2\sinh(2\beta\epsilon)}$$

$$= e^{\beta\epsilon}\left[\cosh(\beta\mu B) \pm \sqrt{\cosh^2(\beta\mu B) - 2\, e^{-2\beta\epsilon}\sinh(2\beta\epsilon)}\,\right] \qquad (7.82)$$

Since all parameters in (7.82) are positive,

$$\lambda_+ > \lambda_-$$

and (7.81a) becomes

$$Z_N(T, B) = \lambda_+^N \left[1 + \left(\frac{\lambda_-}{\lambda_+}\right)^N\right] \qquad (7.81)$$

$$\xrightarrow[N\to\infty]{} \lambda_+^N \qquad (7.81a)$$

In the thermodynamic limit, the Gibbs free energy per site is

$$g(T, B) = \lim_{N\to\infty} \frac{1}{N} G_N(T, B)$$

$$= \lim_{N\to\infty} \frac{1}{N}\left[-k_B T \ln Z_N(T, B)\right]$$

$$= -k_B T \ln \lambda_+ \qquad [\ (7.81a)\ \text{used.}\ ] \qquad (7.83)$$

$$= -\epsilon - k_B T \ln\left[\cosh(\beta\mu B) + \sqrt{\cosh^2(\beta\mu B) - 2\, e^{-2\beta\epsilon}\sinh(2\beta\epsilon)}\,\right]$$

$$[\ (7.82)\ \text{used.}\ ] \qquad (7.84)$$

The average magnetic moment per site is

$$\mu\langle s\rangle = -\left(\frac{\partial g}{\partial B}\right)_T$$

$$= k_B T \frac{\beta\mu \sinh(\beta\mu B) + \dfrac{\beta\mu\cosh(\beta\mu B)\sinh(\beta\mu B)}{\sqrt{\cosh^2(\beta\mu B) - 2\, e^{-2\beta\epsilon}\sinh(2\beta\epsilon)}}}{\cosh(\beta\mu B) + \sqrt{\cosh^2(\beta\mu B) - 2\, e^{-2\beta\epsilon}\sinh(2\beta\epsilon)}}$$

$$= \mu \frac{\sinh(\beta\mu B)}{\sqrt{\cosh^2(\beta\mu B) - 2\, e^{-2\beta\epsilon}\sinh(2\beta\epsilon)}} \qquad (7.85a)$$

Either $\mu\langle s\rangle$ or $\langle s\rangle$ can be chosen as the order parameter for the phase transition.

However, (7.85a) gives

$$\langle s\rangle = 0 \qquad \text{for} \quad B = 0$$

so that there is actually no ferromagnetic state at zero field for any T. Hence, there is no phase transition for the 1-D Ising lattice at any T.

7.F.2. Mean Field Theory for a *d*-D Lattice

For a *d*-D Ising lattice, the Hamiltonian (7.75) can be written as

$$H = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j\in\rho_i} \epsilon\, s_i s_j - \mu B \sum_{i=1}^{N} s_i \qquad (7.86a)$$

where ρ_i is the set of site labels for the nearest neighbors of site (or spin) i. The $\frac{1}{2}$ factor is required since each pair of interacting spins is counted twice in the double sums.

In a mean field approximation, interactions between particles are replaced by their averages. For (7.86a), this means setting

$$\sum_{j \in \rho_i} s_j = \nu \langle s \rangle \qquad \forall\ i \qquad (7.86b)$$

where ν is the number of nearest neighbors and $\langle s \rangle$ the average spin. (7.86a) thus becomes

$$H = -\frac{1}{2} \sum_{i=1}^{N} \nu\, \epsilon \langle s \rangle s_i - \mu B \sum_{i=1}^{N} s_i$$

$$= -\sum_{i=1}^{N} E(B)\, s_i \qquad (7.86)$$

where

$$E(B) = \frac{1}{2} \nu\, \epsilon \langle s \rangle + \mu B \qquad (7.86c)$$

Since (7.86) describes a set of non-interacting spins, the corresponding partition function is simply

$$Z_N = (Z_1)^N = \left(\sum_{s=\pm 1} e^{\beta E s} \right)^N = \left(e^{\beta E} + e^{-\beta E} \right)^N = \left[2 \cosh(\beta E) \right]^N \qquad (7.87)$$

In the thermodynamic limit, the Gibbs free energy per site is

$$g(T, B) = \lim_{N \to \infty} \frac{1}{N} \left(-k_B T \ln Z_N \right)$$

$$= -k_B T \ln\left[2 \cosh(\beta E) \right] \qquad (7.88)$$

The probability of site i having spin s_i is

$$P(s_i) = \frac{1}{Z_1} e^{\beta E s_i} = \frac{1}{2 \cosh(\beta E)} e^{\beta E s_i} \qquad [\ (7.87) \text{ used.}\ ] \qquad (7.89)$$

Thus,

$$\langle s \rangle = \langle s_i \rangle = \sum_{s_i = \pm 1} s_i P(s_i) = \frac{1}{2 \cosh(\beta E)} \left(e^{\beta \epsilon} - e^{-\beta \epsilon} \right) = \tanh(\beta E)$$

$$= \tanh\left[\beta \left(\frac{1}{2} \nu\, \epsilon \langle s \rangle + \mu B \right) \right] \qquad [\ (7.86c) \text{ used.}\ ] \qquad (7.91)$$

which is a self-consistent equation for $\langle s \rangle$.

The solution of (7.91) then gives the magnetization of the lattice as

$$M = N \mu \langle s \rangle \qquad (7.90)$$

For $B = 0$, (7.91) reduces to

$$\langle s \rangle = \tanh\left[\frac{1}{2} \nu \beta \epsilon \langle s \rangle \right] \qquad (7.92)$$

$$= \tanh\left(\alpha \langle s \rangle \right) \qquad \left[\ \alpha = \frac{1}{2} \nu \beta \epsilon = \frac{\nu \epsilon}{2 k_B T}\ \right] \qquad (7.92a)$$

Besides the obviously solution

$$\langle s \rangle = 0 \qquad (7.92b)$$

other solutions of (7.92a) can be solved graphically, with the solutions being the intersects of the curves

$$f_1\left(\langle s \rangle \right) = \langle s \rangle \qquad f_2\left(\langle s \rangle \right) = \tanh\left(\alpha \langle s \rangle \right) \qquad (7.92c)$$

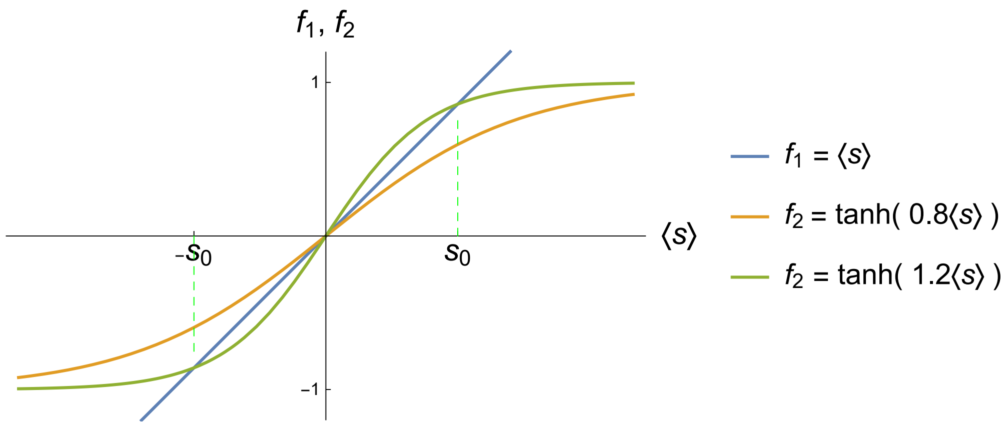

Fig.7.6. Plots of f_1 & f_2 for $\alpha = 0.8$ & 1.2.

As shown in Fig.7.6, there are

1 intersect if $f_2'(0) < f_1'(0)$

3 intersects if $f_2'(0) > f_1'(0)$

where f' is the derivative (or slope) of f.

Using

$$f_1'(0) = 1 \qquad\qquad f_2'(0) = \alpha\,\mathrm{sech}^2(0) = \alpha$$

we have

$$\langle s \rangle = \begin{cases} 0 & \text{if } \alpha < 1 \\ 0, \pm s_0 & \text{if } \alpha > 1 \end{cases}$$

$$= \begin{cases} 0 & \text{if } T > T_C \\ 0, \pm s_0 & \text{if } T < T_C \end{cases} \qquad \text{(7.92a) used.]} \qquad (7.92d)$$

where the critical temperature is defined as

$$T_C = \frac{\nu\,\epsilon}{2\,k_B} \qquad (7.92e)$$

In general, s_0 must be obtained numerically. The result is shown in Fig.7.7.

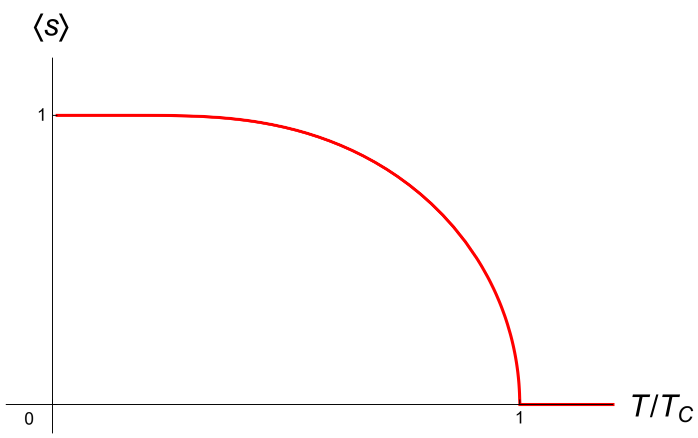

Fig.7.7. Plot of $\langle s \rangle$ vs the reduced temperature T/T_C.

Thus, there is phase transition at zero field for all dimensions, which, in the 1-D case, contradicts with the exact result obtained in the last section.

Putting (7.92d) into (7.86c) & (7.88) gives

$$E(0) = \begin{cases} 0 & \text{if } T > T_C \\ \pm \frac{1}{2} \nu \epsilon s_0 = \pm k_B T_C s_0 & \text{if } T < T_C \end{cases} \tag{7.93a}$$

$$g(T, 0) = \begin{cases} -k_B T \ln 2 & \text{if } T > T_C \\ -k_B T \ln\left[2 \cosh\left(\frac{1}{2} \nu \beta \epsilon s_0\right)\right] & \text{if } T < T_C \end{cases} \tag{7.93}$$

$$= \begin{cases} -k_B T \ln 2 & \text{if } T > T_C \\ -k_B T \ln\left[2 \cosh\left(\frac{T_C}{T} s_0\right)\right] & \text{if } T < T_C \end{cases} \tag{7.93b}$$

The internal energy is [see (7.64)]

$$U = -\frac{\partial \ln Z_N}{\partial \beta}$$

$$= -N \frac{\sinh(\beta E)}{\cosh(\beta E)} \left(E + \beta \frac{\partial E}{\partial \beta}\right) \qquad \text{[(7.87) used.]}$$

$$= -N \langle s \rangle \left(\frac{1}{2} \nu \epsilon \langle s \rangle + \mu B + \frac{1}{2} \nu \epsilon \beta \frac{\partial \langle s \rangle}{\partial \beta}\right) \qquad \text{[(7.92) \& (7.86c) used.]} \tag{7.94a}$$

On the other hand, taking the average of (7.86) gives

$$U = \langle H \rangle = -N E(B) \langle s \rangle$$

$$= -N \langle s \rangle \left(\frac{1}{2} \nu \epsilon \langle s \rangle + \mu B\right) \tag{7.94}$$

The discrepancy between (7.94 & a) is the price to pay for the mean field approximation. Following Reichl, we shall take (7.94) as the expression for U.

The heat capacity is

$$C_N = \left(\frac{\partial U}{\partial T}\right)_N = \frac{d\beta}{dT} \left(\frac{\partial U}{\partial \beta}\right)_N = -k_B \beta^2 \left(\frac{\partial U}{\partial \beta}\right)_N \tag{7.95a}$$

where

$$\frac{d\beta}{dT} = -\frac{1}{k_B T^2} = -k_B \beta^2$$

At zero field, (7.94) gives

$$C_N = N \nu \epsilon k_B \beta^2 \langle s \rangle \frac{\partial \langle s \rangle}{\partial \beta} \tag{7.95}$$

From (7.92), we have

$$\frac{\partial \langle s \rangle}{\partial \beta} = \operatorname{sech}^2\left[\frac{1}{2} \nu \beta \epsilon \langle s \rangle\right] \frac{1}{2} \nu \epsilon \left(\langle s \rangle + \beta \frac{\partial \langle s \rangle}{\partial \beta}\right) \tag{7.96}$$

$$= \frac{\frac{1}{2} \nu \epsilon \langle s \rangle \operatorname{sech}^2\left[\frac{1}{2} \nu \beta \epsilon \langle s \rangle\right]}{1 - \frac{1}{2} \nu \beta \epsilon \operatorname{sech}^2\left[\frac{1}{2} \nu \beta \epsilon \langle s \rangle\right]} \qquad \text{[Obtained by solving (7.96).]}$$

$$= \frac{\nu \epsilon \langle s \rangle}{2 \cosh^2\left[\frac{1}{2} \nu \beta \epsilon \langle s \rangle\right] - \nu \beta \epsilon} \tag{7.97}$$

$$= \frac{k_B T_c \langle s \rangle}{\cosh^2\left[\frac{T_C}{T} \langle s \rangle\right] - \frac{T_C}{T}} \qquad \text{[(7.92e) used.]} \tag{7.97a}$$

Putting (7.97) into (7.95) gives

$$C_N = N k_B \frac{\left(\nu \beta \epsilon \langle s \rangle\right)^2}{2 \cosh^2\left[\frac{1}{2} \nu \beta \epsilon \langle s \rangle\right] - \nu \beta \epsilon} \qquad \text{[(7.97) used.]}$$

$$= N k_B \frac{2\left(\frac{T_C}{T} \langle s \rangle\right)^2}{\cosh^2\left(\frac{T_C}{T} \langle s \rangle\right) - \frac{T_C}{T}} \qquad \text{[(7.92e) used.]} \qquad (7.98)$$

$$= \begin{cases} 0 & \text{for } T > T_C \\ N k_B \dfrac{2\left(\frac{T_C}{T} s_0\right)^2}{\cosh^2\left(\frac{T_C}{T} s_0\right) - \frac{T_C}{T}} & \text{for } T \lesssim T_C \end{cases} \qquad \text{[(7.92d) used.]} \qquad (7.98a)$$

The transition point is therefore a λ-point. The magnitude of the discontinuity is determined as follows.

As we approach the critical from the low T side, $\langle s \rangle \to 0$ so that (7.92a) becomes

$$\langle s \rangle = \tanh\left(\frac{T_C}{T} \langle s \rangle\right) = \frac{T_C}{T} \langle s \rangle - \frac{1}{3}\left(\frac{T_C}{T} \langle s \rangle\right)^3 + O\left(\langle s \rangle^5\right)$$

$\rightarrow \qquad \langle s \rangle = 0$

or $\qquad \frac{1}{3}\left(\frac{T_C}{T} \langle s \rangle\right)^2 = \frac{T_C}{T} - 1 + O\left(\langle s \rangle^4\right)$

$\rightarrow \qquad s_0 = \frac{T}{T_C} \sqrt{3 \tau} + O\left(s_0^2 = \tau\right) \qquad (7.98b)$

where

$$\tau = \frac{T_C}{T} - 1 \qquad (7.98c)$$

(7.98a) then gives

$$\lim_{T \to T_C^-} C_N = \lim_{T \to T_C^-} N k_B \frac{2\left(\frac{T_C}{T} s_0\right)^2}{-\frac{T_C}{T} + 1 + \left(\frac{T_C}{T} s_0\right)^2 + O(s_0^4)}$$

$$= \lim_{T \to T_C^-} 2 N k_B \frac{3 \tau + O\left(\tau^2\right)}{-\frac{T_C}{T} + 1 + 3 \tau + O(\tau^2)} \qquad \text{[(7.98b) used.]}$$

$$= \lim_{T \to T_C^-} 2 N k_B \frac{3 \tau + O(\tau^2)}{2 \tau + O(\tau^2)}$$

$$= 3 N k_B \qquad (7.98d)$$

which is also the magnitude of the discontinuity of C_N at the λ point [see Fig.7.8].

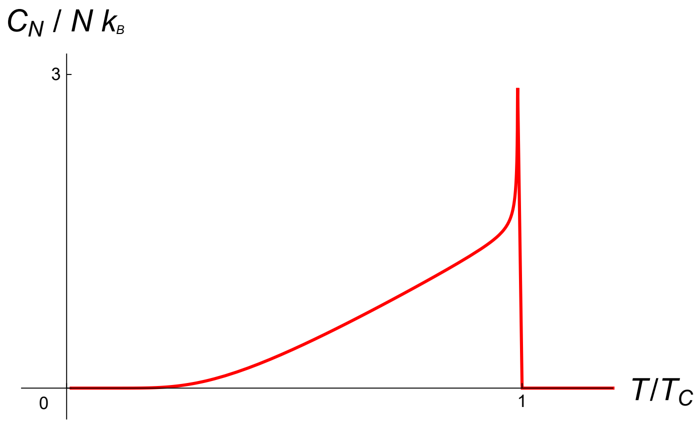

Fig.7.8. Temperature dependence of the heat capacity.

At $B=0$, (7.98a) gives $C_N = 0$ for $T > T_C$. This means the system cannot absorb heat if it is in the disorder phase. Indeed, (7.86c) reduces to

$$E(B) = \mu B \qquad \text{for} \quad \langle s \rangle = 0$$

The only way to change T is then by changing B.

Another quantity of interest is the magnetic susceptibility

$$\chi_{T,N}(B) = \left(\frac{\partial M}{\partial B}\right)_{T,N} = N \mu \left(\frac{\partial \langle s \rangle}{\partial B}\right)_{T,N} \qquad \text{[(7.90) used.]} \qquad (7.99)$$

(7.91) gives

$$\left(\frac{\partial \langle s \rangle}{\partial B}\right)_{T,N} = \operatorname{sech}^2\left[\beta\left(\frac{1}{2}\nu\epsilon\langle s\rangle + \mu B\right)\right] \cdot \beta\left[\frac{1}{2}\nu\epsilon\left(\frac{\partial \langle s \rangle}{\partial B}\right)_{T,N} + \mu\right] \qquad (7.100)$$

$$= \frac{\beta\mu \operatorname{sech}^2\left[\beta\left(\frac{1}{2}\nu\epsilon\langle s\rangle + \mu B\right)\right]}{1 - \frac{1}{2}\nu\beta\epsilon \operatorname{sech}^2\left[\beta\left(\frac{1}{2}\nu\epsilon\langle s\rangle + \mu B\right)\right]} \qquad \text{[Obtained by solving (7.100).]}$$

$$= \frac{\beta\mu}{\cosh^2\left[\beta\left(\frac{1}{2}\nu\epsilon\langle s\rangle + \mu B\right)\right] - \frac{1}{2}\nu\beta\epsilon} \qquad (7.101)$$

(7.99) becomes

$$\chi_{T,N}(B) = \frac{N\beta\mu^2}{\cosh^2\left[\beta\left(\frac{1}{2}\nu\epsilon\langle s\rangle + \mu B\right)\right] - \frac{1}{2}\nu\beta\epsilon} \qquad (7.102)$$

$$= \frac{\frac{N\mu^2}{k_B T}}{\cosh^2\left[\frac{T_C}{T}\langle s\rangle + \frac{\mu B}{k_B T}\right)\right] - \frac{T_C}{T}}$$

$$= \frac{2N\mu^2}{\nu\epsilon} \frac{\frac{T_C}{T}}{\cosh^2\left[\frac{T_C}{T}\langle s\rangle + \frac{\mu B}{k_B T}\right)\right] - \frac{T_C}{T}} \qquad (7.102a)$$

$$\to \quad \chi_{T,N}(0) = \frac{2N\mu^2}{\nu\epsilon} \frac{\frac{T_C}{T}}{\cosh^2\left(\frac{T_C}{T}\langle s\rangle\right) - \frac{T_C}{T}} \qquad (7.103)$$

For $T \approx T_C$, $\langle s \rangle \to 0$ and we have

$$\chi_{T,N}(0) = \frac{2 N \mu^2}{\nu \epsilon} \frac{\frac{T_C}{T}}{-\frac{T_C}{T} + 1 + \left(\frac{T_C}{T} s_0\right)^2 + O(s_0^4)}$$

$$= \frac{2 N \mu^2}{\nu \epsilon} \frac{\frac{T_C}{T}}{-\tau + 3\tau + O(\tau^2)} \qquad \text{[(7.98b \&c) used.]}$$

$$\approx \frac{N \mu^2}{\nu \epsilon} \frac{1}{\tau} \qquad \text{for} \quad \tau \to 0$$

which corresponds to the critical exponents $\gamma = \gamma' = 1$ [see (8.43) of §8.C.2 for definition of γ].

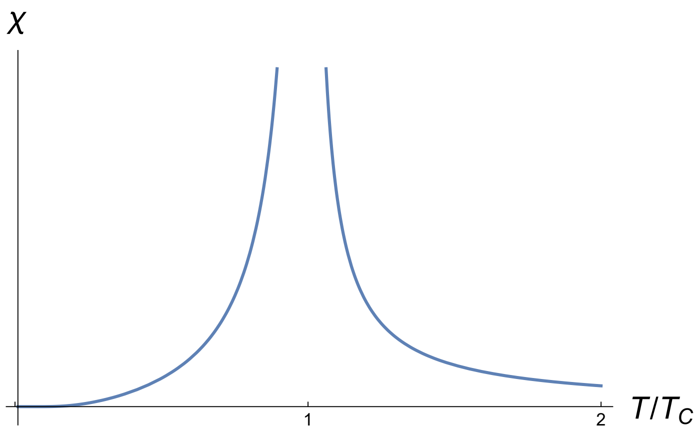

Fig.7.9. Plot of $\chi_{T,N}(0)$ vs the reduced temperature T/T_C.

Code

```
In[∘]:= ss = s /. NSolve[s == Tanh[1.5 s], s, Reals]
Out[∘]= {-0.85856, 0., 0.85856}

In[∘]:= (* Fig.7.6 *)
Plot[{s, Tanh[.8 s], Tanh[1.5 s]}, {s, -2, 2},
        PlotRange → {All, 1.2 {-1, 1}},
        AxesLabel → {"⟨s⟩", "f₁, f₂"},
        AspectRatio → Automatic,
        Ticks → {{{ss[[1]], "-S₀"}, {ss[[3]], "S₀"}}, {-1, 0, 1}},
        PlotLegends → { "f₁ = ⟨s⟩", "f₂ = tanh( 0.8⟨s⟩ )", "f₂ = tanh( 1.2⟨s⟩ )"},
        Epilog → {Dashed, Green, Line[{{ss[[1]], 0}, {ss[[1]], Tanh[1.2 ss[[1]]]}}],
                Line[{{ss[[3]], 0}, {ss[[3]], Tanh[1.2 ss[[3]]]}}]}
]

In[∘]:= avs[T_] := s /. NSolve[s == Tanh[s/T], s, Reals][[1]] // Abs
```

```
In[•]:= (* Fig.7.6 *)
        Plot[avs[T], {T, 0.01, 1.2},
             PlotRange → {{-.1, 1.2}, {-.1, 1.2}},
             PlotStyle → Red,
             AxesLabel → {"T/TC", "⟨s⟩"},
             Ticks → {{0, 1}, {0, 1}},
             Epilog → Text["0", -.05 {1, 1}]
            ]

In[•]:= davs[T_, dT_] := (avs[T + dT] - avs[T]) / dT

In[•]:= cn[T_, dT_] := -avs[T] × davs[T, dT]

In[•]:= Plot[cn[T, .01], {T, .01, 1.2},
             PlotRange → {{-.1, 1.2}, {-.3, 3.2}},
             PlotStyle → Red,
             AxesLabel → {"T/TC", "CN / N kB"},
             Ticks → {{0, 1}, {0, 3}},
             Epilog → Text["0", -.05 {1, 3}]
        ]

In[•]:= χ[T_] := 1 / (T Cosh[1/T avs[T]] - 1)

In[•]:= (* Fig.7.9 *)
        Plot[χ[T], {T, 0.01, 2},
             AxesLabel → {"T/TC", "χ"},
             Ticks → {{0, 1, 2}, None}
        ]
```