7.H.2. Fermi-Dirac Ideal Gases

For an ideal gas of spin $s = \frac{1}{2}$ fermions, the fermionic grand partition function (7.123) becomes

$$Z_{FD}(T, V, \mu') = \prod_l \left[\left(\sum_{n_{l\uparrow}=0}^{1} e^{-\beta n_{l\uparrow}(\epsilon_l - \mu')} \right) \left(\sum_{n_{l\downarrow}=0}^{1} e^{-\beta n_{l\downarrow}(\epsilon_l - \mu')} \right) \right] \qquad (7.157a)$$

where the subscripts ↑ & ↓ denote $\frac{s_z}{\hbar} = \frac{1}{2}$ & $-\frac{1}{2}$, respectively. Writing out the sums explicitly, we get

$$Z_{FD}(T, V, \mu') = \prod_l \left[1 + e^{-\beta(\epsilon_l - \mu')} \right]^2 \qquad (7.157)$$

which is easily generalized to the case of spin s fermions as

$$Z_{FD}(T, V, \mu') = \prod_l \left[1 + e^{-\beta(\epsilon_l - \mu')} \right]^g \qquad g = 2s + 1 \qquad (7.158)$$

The corresponding grand potential is [c.f. (7.125) of §7.H.1]

$$\begin{aligned} \Omega_{FD}(T, V, \mu') &= -k_B T \ln Z_{FD}(T, V, \mu') \\ &= -g\, k_B T \sum_l \ln\left[1 + e^{-\beta(\epsilon_l - \mu')} \right] \end{aligned} \qquad (7.159)$$

so that [c.f. (7.126)]

$$\begin{aligned} \langle N \rangle &= -\left(\frac{\partial\, \Omega_{FD}}{\partial\, \mu'} \right)_{T\,V} \\ &= \sum_l \frac{g}{e^{\beta(\epsilon_l - \mu')} + 1} \\ &= \sum_l \langle n_l \rangle \end{aligned} \qquad (7.160)$$

where

$$\langle n_l \rangle = \frac{g}{e^{\beta(\epsilon_l - \mu')} + 1} = \text{occupation number of state } l. \qquad (7.161a)$$

$$= \frac{g}{\frac{1}{z} e^{\beta \epsilon_l} + 1} \qquad z = e^{\beta \mu'} \qquad (7.161)$$

Since

$$0 \le \langle n_l \rangle \le g$$

we have

$$\frac{1}{z} e^{\beta \epsilon_l} + 1 \ge 1 \qquad \to \qquad \frac{1}{z} e^{\beta \epsilon_l} \ge 0$$

Since $e^{\beta \epsilon_l} > 0$, this means

$$z > 0 \qquad (7.161b)$$

but there is no restriction on μ', which is why we did not follow the example of bosons and use ϵ_0 as the reference point.

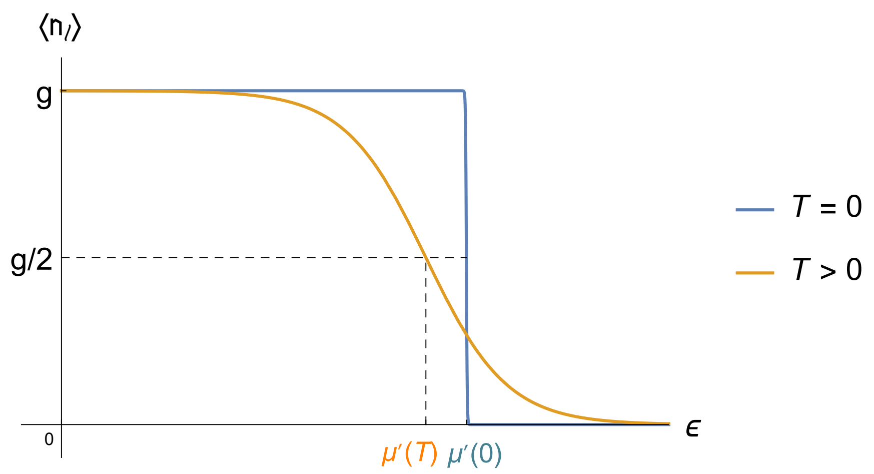

Fig.7.16. Plots of $\langle \mathfrak{n}_l \rangle$ as a function ϵ for $T = 0$ & $T > 0$.
For the sake of clarity, the temperature dependence of μ' is greatly exaggerated.

Plots of the occupation number $\langle \mathfrak{n}_l \rangle$ as a function of energy ϵ are shown in Fig.7.16, with an exaggerated representation of the effect of T on μ'. The most striking feature is that, at $T = 0$, all states with energy below (above) $\mu'(0)$ are occupied (unoccupied). Thus, if we add energy ϵ to the system, only particles with energy an amount ϵ or less below $\mu'(0)$ can absorb it and change to a new state. This behavior is akin to the sea where only water on or near the surface can be affected by the wind. The fermion distribution, for any T, is therefore often called the Fermi sea and $\mu'(0)$ the Fermi energy ϵ_F. In solid state physics, even $\mu'(T)$ is often called the Fermi energy.

For $T > 0$, we have

$$\langle \mathfrak{n}_l \rangle = \begin{cases} g & \text{for} \quad \epsilon << \mu' \\ g/2 & \text{at} \quad \epsilon = \mu' \\ g\, e^{-\beta(\epsilon-\mu')} & \text{for} \quad \epsilon >> \mu' \end{cases} \tag{7.161c}$$

which can be easily verified using (7.161).

Using the familiar approximation for free particles (plane-wave wavefunctions)

$$\sum_l \approx \frac{V}{(2\pi)^3} \int d\Omega \int_0^\infty dk\, k^2 \tag{7.162}$$

the grand potential (7.159) becomes

$$\Omega_{FD}(T, V, \mu') = -g\, k_B T \frac{4\pi V}{(2\pi)^3} \int_0^\infty dk\, k^2 \ln\left[1 + z\, e^{-\beta \hbar^2 k^2/2m}\right] \tag{7.163}$$

while (7.160-1) give the average number of particles as

$$\langle N \rangle = \frac{4\pi V}{(2\pi)^3} \int_0^\infty dk\, k^2 \frac{g}{\frac{1}{z} e^{\beta \hbar^2 k^2/2m} + 1} \tag{7.164}$$

As in the bosonic case, we set

$$x^2 = \beta \frac{\hbar^2 k^2}{2m} = \frac{\lambda_T^2}{4\pi} k^2 \tag{7.164a}$$

so that

$$\int_0^\infty dk\, k^2 \ln\left[1 + z\, e^{-\beta \hbar^2 k^2/2m}\right] = \frac{(4\pi)^{3/2}}{\lambda_T^3} \int_0^\infty dx\, x^2 \ln\left[1 + z\, e^{-x^2}\right]$$

$$= \frac{2\pi^2}{\lambda_T^3} f_{5/2}(z) \tag{7.166a}$$

where

$$f_{5/2}(z) = \frac{4}{\sqrt{\pi}} \int_0^\infty dx\, x^2 \ln\left[1 + z\, e^{-x^2}\right] \tag{7.166b}$$

$$= \frac{4}{\sqrt{\pi}} \int_0^\infty dx\, x^2 \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n} \left(z\, e^{-x^2}\right)^n$$

$$= \frac{4}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n} z^n \int_0^\infty dx\, x^2\; e^{-n x^2}$$

$$= \frac{4}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n} z^n \frac{1}{2\, n^{3/2}} \int_0^\infty dy \sqrt{y}\; e^{-y} \qquad x = \sqrt{\frac{y}{n}}$$

$$= \frac{2}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n^{5/2}} z^n\, \Gamma\left(\frac{3}{2}\right)$$

$$= \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n^{5/2}} z^n \tag{7.166}$$

(7.163) thus becomes

$$\Omega_{\text{FD}}(T, V, \mu') = -g\, k_B\, T \frac{V}{\lambda_T^3} f_{5/2}(z) \tag{7.165a}$$

so that the pressure of the Fermi gas is

$$P = -\frac{\Omega_{\text{FD}}}{V} = \frac{g\, k_B\, T}{\lambda_T^3} f_{5/2}(z) \tag{7.165}$$

Similarly, the integral in (7.164) becomes

$$\int_0^\infty dk\, k^2 \frac{1}{\frac{1}{z} e^{\beta \hbar^2 k^2 / 2m} + 1} = \frac{(4\pi)^{3/2}}{\lambda_T^3} \int_0^\infty dx\, x^2 \frac{1}{\frac{1}{z} e^{x^2} + 1}$$

$$= \frac{2\pi^2}{\lambda_T^3} f_{3/2}(z) \tag{7.167a}$$

where

$$f_{3/2}(z) = \frac{4}{\sqrt{\pi}} \int_0^\infty dx\, x^2 \frac{1}{\frac{1}{z} e^{x^2} + 1} \tag{7.168a}$$

$$= \frac{4}{\sqrt{\pi}} \int_0^\infty dx\, x^2\; z\, e^{-x^2} \frac{1}{1 + z\, e^{-x^2}}$$

$$= \frac{4}{\sqrt{\pi}} \int_0^\infty dx\, x^2\, z\, e^{-x^2} \sum_{n=0}^{\infty} (-)^n \left(z\, e^{-x^2}\right)^n$$

$$= \frac{4}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-)^n z^{n+1} \int_0^\infty dx\, x^2\; e^{-(n+1)x^2}$$

$$= \frac{4}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-)^n z^{n+1} \frac{1}{2\,(n+1)^{3/2}} \int_0^\infty dy \sqrt{y}\; e^{-y} \qquad x = \sqrt{\frac{y}{n+1}}$$

$$= \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-)^n}{(n+1)^{3/2}} z^{n+1}\, \Gamma\left(\frac{3}{2}\right)$$

$$= \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n^{3/2}} z^n \tag{7.168}$$

(7.164) can therefore be written as

$$\overline{n} = \langle n \rangle = \frac{\langle N \rangle}{V} = \frac{g}{\lambda_T^3} f_{3/2}(z) \tag{7.167}$$

where $\overline{n} = \langle n \rangle$ since the system is always in the gaseous phase.

The Fermi integrals are defined as

$$F_j(\mu) = \frac{1}{\Gamma(j+1)} \int_0^\infty dx \, \frac{x^j}{e^{x-\mu}+1} = \frac{1}{\Gamma(j+1)} \int_0^\infty dx \, \frac{x^j}{\frac{1}{z} e^x + 1}$$

$$= -\mathrm{Li}_{j+1}(-e^\mu) = -\mathrm{Li}_{j+1}(-z) \qquad (7.168b)$$

where $\mathrm{Li}_s(z)$ is the polylogarithm function defined as

$$\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{1}{n^s} z^n \qquad (7.168c)$$

which is a generalization of the Zeta function $\zeta(s)$.

Hence,

$$f_{3/2}(z) = -\mathrm{Li}_{3/2}(-z) = F_{1/2}(\beta\mu')$$

$$f_{5/2}(z) = -\mathrm{Li}_{5/2}(-z) = F_{3/2}(\beta\mu') \qquad (7.168d)$$

Note: The bosonic counterparts of the Fermi integrals are called Bose integrals. They are also related to the polylogarithms [see §7.H.1].

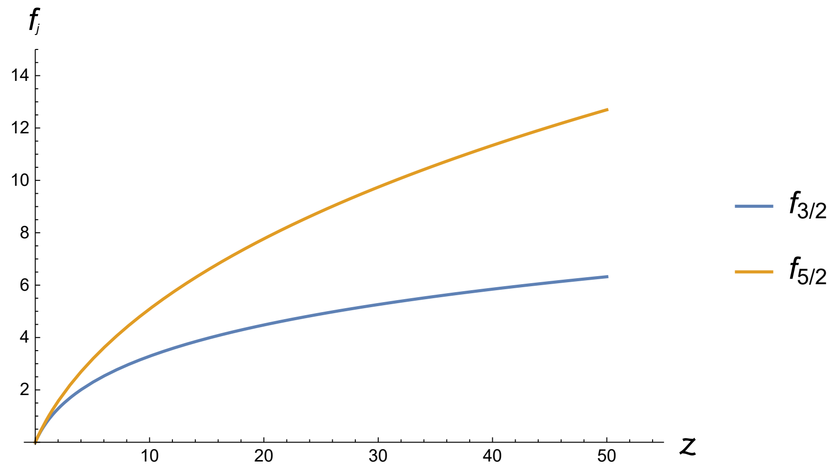

Fig.7.17. Plots of $f_{3/2}(z)$ & $f_{5/2}(z)$.

The volume per particle $v = \frac{1}{\overline{n}} = \frac{1}{\langle n \rangle}$ is related to z by [see (7.167)]

$$\frac{1}{v} = \frac{g}{\lambda_T^3} f_{3/2}(z) \qquad (7.169a)$$

which can be used to write (7.165) as

$$P = \frac{k_B T}{v} \frac{f_{5/2}(z)}{f_{3/2}(z)} \qquad (7.169b)$$

where z can be solved numerically from (7.169a) for given T & v.

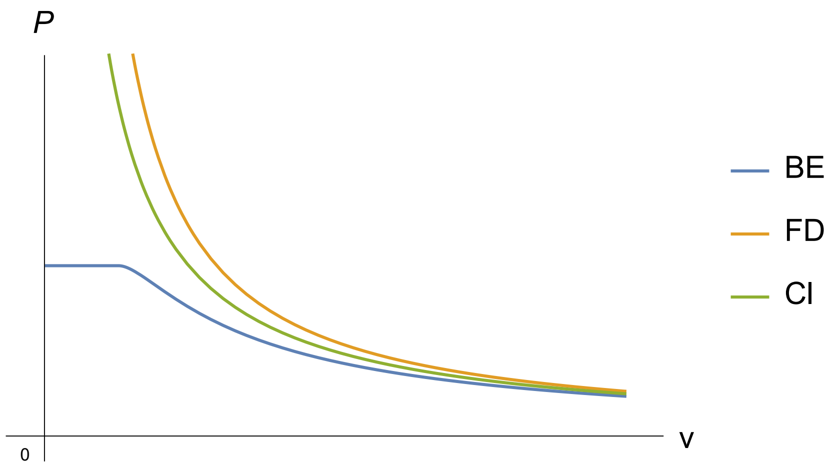

Fig.7.18. Isotherms of the same T for the Bose (BE), Fermi (FD) & classical (Cl) version of the same ideal gas, spin degree of freedom being ignored. All 3 versions coincide in the classical limit of $v \to \infty$, as expected.

From the isotherms shown in Fig.7.18, we see that, for a given T,

$$P_{FD} > P_{Cl} > P_{BE} \qquad \forall\, v$$

which means there is an effective force of quantum origin that is repulsive & attractive for fermions & bosons, respectively. Obviously, this "quantum force" arises from the statistics dictated by the symmetry under particle exchange. Therefore, it is called the exchange force.

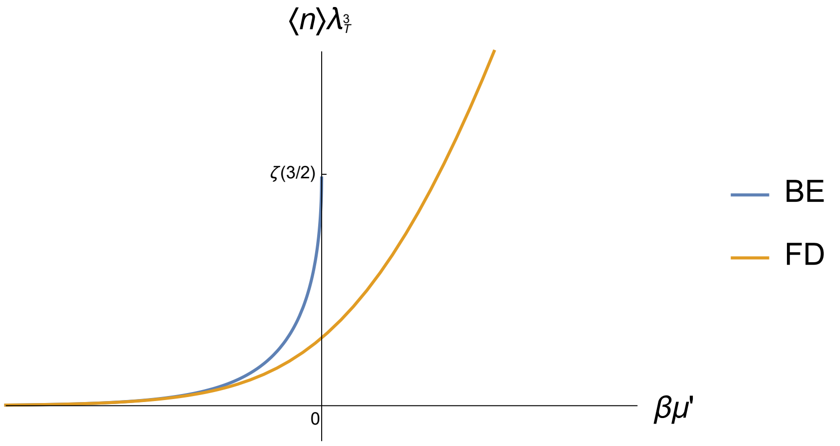

Fig.7.19. Plots of $\langle n \rangle \lambda_T^3$ vs $\beta\mu'$.

Fig.7.19 shows the plots of $\langle n \rangle \lambda_T^3$ vs $\beta\mu'$ using (7.147a) & (7.169a) for the Bose & Fermi gases, respectively, assuming $g = 1$. Since $\mu' \le \epsilon_0 = 0$ for free bosons, the curve stops at $\beta\mu' = 0$, where

$$\langle n \rangle \lambda_T^3 \Big|_{\beta\mu'=0} = \begin{cases} g_{3/2}(1) = \zeta(3/2) & \text{for bosons} \\ f_{3/2}(1) = \left(1 - \dfrac{1}{\sqrt{2}}\right)\zeta(3/2) & \text{for fermions} \end{cases}$$

Consider now (7.169a) in the form

$$\frac{\langle n \rangle \lambda_T^3}{g} = f_{3/2}(z) = \sum_{n=1}^{\infty} (-)^{n+1} \frac{z^n}{n^{3/2}} \qquad (7.169c)$$

We can invert (7.169c) by setting

$$z = \sum_{m=1}^{\infty} c_m \eta^m \qquad \eta = \frac{\langle n \rangle \lambda_T^3}{g} \qquad (7.169d)$$

Putting (7.169d) back into (7.169c), we get

$$\eta = \sum_{n=1}^{\infty} (-)^{n+1} \frac{1}{n^{3/2}} \left(\sum_{m=1}^{\infty} c_m \eta^m \right)^n$$

$$= \sum_{m=1}^{\infty} c_m \eta^m - \frac{1}{2^{3/2}} \left(\sum_{m=1}^{\infty} c_m \eta^m \right)^2 + \frac{1}{3^{3/2}} \left(\sum_{m=1}^{\infty} c_m \eta^m \right)^3 + \ldots$$

$$= c_1 \eta + c_2 \eta^2 + c_3 \eta^3 + \ldots - \frac{1}{2^{3/2}} \left(c_1^2 \eta^2 + 2 c_1 c_2 \eta^3 + \ldots \right) + \frac{1}{3^{3/2}} \left(c_1^3 \eta^3 + \ldots \right) + O\left(\eta^4\right)$$

(7.169e)

Since (7.169e) is valid for all η, the coefficient of each power of η must vanish separately, i.e.,

$$1 = c_1$$

$$0 = c_2 - \frac{1}{2^{3/2}} c_1^2$$

$$0 = c_3 - \frac{2}{2^{3/2}} c_1 c_2 + \frac{1}{3^{3/2}} c_1^3$$

$$\vdots$$

where the n^{th} equation gives c_n as a linear combination of c_m's with $m < n$. The set of equations can therefore be solved iteratively starting from the top, giving

$$c_1 = 1$$

$$c_2 = \frac{1}{2^{3/2}}$$

$$c_3 = \frac{2}{2^{3/2}} \frac{1}{2^{3/2}} - \frac{1}{3^{3/2}} = \frac{1}{2^2} - \frac{1}{3^{3/2}}$$

$$\vdots$$

(7.169d) then becomes

$$z = \eta + \frac{1}{2^{3/2}} \eta^2 + \left(\frac{1}{2^2} - \frac{1}{3^{3/2}} \right) \eta^3 + \ldots$$

$$= \frac{\langle n \rangle \lambda_T^3}{g} + \frac{1}{2^{3/2}} \left(\frac{\langle n \rangle \lambda_T^3}{g} \right)^2 + \left(\frac{1}{2^2} - \frac{1}{3^{3/2}} \right) \left(\frac{\langle n \rangle \lambda_T^3}{g} \right)^3 + \ldots \qquad (7.169)$$

For $T \to \infty$,

$$\eta = \frac{\langle n \rangle \lambda_T^3}{g} \propto T^{-3/2} \to 0$$

so that (7.169) implies

$$e^{\beta \mu'} = z \to 0$$

which means

$$\beta \mu' \to -\infty$$

Since $\beta \to 0$, we must have

$$\mu' \to -\infty \qquad \text{for} \qquad T \to \infty$$

For $T \to 0$,

$$\eta \to \infty$$

so that (7.169) implies

$$z \to \infty \qquad \Rightarrow \qquad \beta \mu' \to \infty$$

Since $\beta \to \infty$, it is possible that μ' remain finite as $T \to 0$. (7.169f)

To find out if this is indeed the case, we need to obtain the limit of (7.169c) as $T \to 0$. Using (7.168a) on (7.169c), we have

$$\frac{\langle n\rangle \lambda_T^3}{g} = f_{3/2}(z) = \frac{4}{\sqrt{\pi}} \int_0^\infty d\,x\,x^2 \frac{1}{\frac{1}{z} e^{x^2} + 1} \qquad x^2 = \beta \frac{\hbar^2 k^2}{2\,m}$$

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty d\,y \sqrt{y} \frac{1}{\frac{1}{z} e^{y} + 1} \qquad y = x^2 \qquad x = \sqrt{y}$$

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty d\,y \sqrt{y} \frac{1}{e^{y-\nu} + 1} \qquad \nu = \beta\,\mu' \qquad (7.170a)$$

Using

$$d \frac{y^{3/2}}{e^{y-\nu}+1} = \left[\frac{3}{2} \frac{\sqrt{y}}{e^{y-\nu}+1} - \frac{y^{3/2}\,e^{y-\nu}}{(e^{y-\nu}+1)^2}\right] d\,y$$

we can integrate by part (7.170a) to get

$$f_{3/2}(z) = \frac{4}{3\sqrt{\pi}} \left[\frac{y^{3/2}}{e^{y-\nu}+1} \Big|_0^\infty + \int_0^\infty d\,y \frac{y^{3/2}\,e^{y-\nu}}{(e^{y-\nu}+1)^2} \right]$$

$$= \frac{4}{3\sqrt{\pi}} \int_0^\infty d\,y \frac{y^{3/2}\,e^{y-\nu}}{(e^{y-\nu}+1)^2} \qquad (7.170)$$

Consider the function

$$\Delta(y,\nu) \equiv \frac{e^{y-\nu}}{(e^{y-\nu}+1)^2} \qquad (7.170b)$$

$$= -\frac{d}{d\,y}\left(\frac{1}{e^{y-\nu}+1}\right)$$

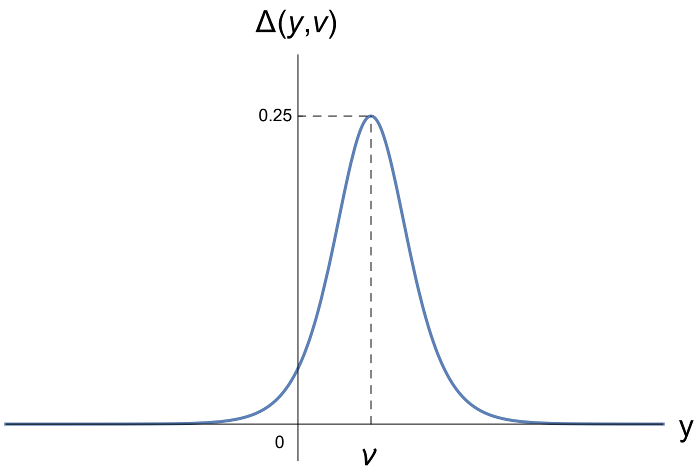

Fig.7.19. Plot of $\Delta(y,\nu)$.

Its peak is at $y = \nu$ with $\Delta(\nu,\nu) = \frac{1}{4}$. Its half-width positions are the solutions of

$$\frac{x}{(x+1)^2} = \frac{1}{8} \qquad x = e^{y-\nu}$$

$$\rightarrow \quad x^2 - 6\,x + 1 = 0$$

$$\therefore \quad x = 3 \pm 2\sqrt{2} \qquad \rightarrow \qquad y_\pm = \nu + \ln\left(3 \pm 2\sqrt{2}\right)$$

The half-width of $\Delta(y,\nu)$ is therefore

$$w = y_+ - y_- = \frac{\ln\left(3+2\sqrt{2}\right)}{\ln\left(3-2\sqrt{2}\right)} \qquad (7.170c)$$

which is independent of v.

This suggests writing (7.170) as

$$f_{3/2}(z) = \frac{4}{3\sqrt{\pi}} \int_{-v}^{\infty} dt \frac{(t+v)^{3/2} e^t}{(e^t+1)^2} \qquad t = y - v \qquad (7.171a)$$

where the integrand vanishes a few half-width w's from the origin. Hence, for $v > w$, we can write

$$\begin{aligned} f_{3/2}(z) &= \frac{4}{3\sqrt{\pi}} \int_{-v}^{\infty} dt \frac{v^{3/2} e^t}{(e^t+1)^2} \left(1 + \frac{t}{v}\right)^{3/2} \\ &= \frac{4}{3\sqrt{\pi}} \int_{-v}^{\infty} dt \frac{v^{3/2} e^t}{(e^t+1)^2} \left[1 + \frac{3t}{2v} + \frac{3}{8}\left(\frac{t}{v}\right)^2 + \ldots\right] \\ &= \frac{4}{3\sqrt{\pi}} \int_{-v}^{\infty} dt \frac{e^t}{(e^t+1)^2} \left(v^{3/2} + \frac{3}{2} t v^{1/2} + \frac{3}{8} t^2 v^{-1/2} + \ldots\right) \qquad (7.171) \end{aligned}$$

At low temperatures such that $v = \beta\mu' \gg w$, we can write (7.171) as

$$f_{3/2}(z) = \frac{4}{3\sqrt{\pi}} \int_{-\infty}^{\infty} dt \frac{e^t}{(e^t+1)^2} \left(v^{3/2} + \frac{3}{2} t v^{1/2} + \frac{3}{8} t^2 v^{-1/2} + \ldots\right) \qquad (7.172)$$

which is a linear combination of integrals of the form

$$\begin{aligned} I_n &= \int_{-\infty}^{\infty} dt \frac{e^t}{(e^t+1)^2} t^n \qquad n = 1, 2, 3, \ldots \qquad (7.173) \\ &= \int_{-\infty}^{0} dt \frac{e^t}{(e^t+1)^2} t^n + \int_{0}^{\infty} dt \frac{e^t}{(e^t+1)^2} t^n \\ &= \int_{0}^{\infty} dt \frac{e^{-t}}{(e^{-t}+1)^2} (-t)^n + \int_{0}^{\infty} dt \frac{e^{-t}}{(1+e^{-t})^2} t^n \end{aligned}$$

Thus,

$$I_n = 0 \qquad \text{for} \qquad n \text{ odd.}$$

For n even, we have

$$\begin{aligned} I_n &= 2\int_0^{\infty} dt\, e^{-t} t^n \left(1 + e^{-t}\right)^{-2} \\ &= 2\sum_{m=0}^{\infty} (-)^m (m+1) \int_0^{\infty} dt\, e^{-t} t^n e^{-mt} \\ &= 2\sum_{m=1}^{\infty} (-)^{m+1} m \int_0^{\infty} dt\, t^n e^{-mt} \\ &= 2\sum_{m=1}^{\infty} \frac{(-)^{m+1}}{m^n} \int_0^{\infty} ds\, s^n e^{-s} \qquad s = mt \\ &= 2\sum_{m=1}^{\infty} \frac{(-)^{m+1}}{m^n} \Gamma(n+1) \\ &= 2(n!)\left[\sum_{m=1}^{\infty} \frac{1}{(2m-1)^n} - \sum_{m=1}^{\infty} \frac{1}{(2m)^n}\right] \qquad [\text{ (Odd } m \text{ sum)} - \text{(even } m \text{ sum). }] \\ &= 2(n!)\left[\sum_{m=1}^{\infty} \frac{1}{m^n} - 2\sum_{m=1}^{\infty} \frac{1}{(2m)^n}\right] \qquad [\text{ (Odd } m \text{ sum)} = \text{(All } m \text{ sum)} - \text{(even } m \text{ sum). }] \\ &= 2(n!)\left(1 - 2^{1-n}\right) \sum_{m=1}^{\infty} \frac{1}{m^n} \\ &= 2(n!)\left(1 - 2^{1-n}\right) \zeta(n) \qquad (7.173a) \end{aligned}$$

where

$$\zeta(0) = -\frac{1}{2} \qquad \zeta(2) = \frac{\pi^2}{6} \qquad \zeta(4) = \frac{\pi^2}{90} \qquad \zeta(4) = \frac{\pi^2}{945}$$

$$\rightarrow \quad I_0 = 1 \qquad I_2 = \frac{\pi^2}{3} \qquad I_4 = \frac{7\,\pi^2}{15} \tag{7.173b}$$

(7.172) thus becomes

$$\begin{aligned}
\frac{\langle n\rangle \lambda_T^3}{g} = f_{3/2}(z) &= \frac{4}{3\sqrt{\pi}}\left(\nu^{3/2} I_0 + \frac{3}{8} I_2 \nu^{-1/2} + \ldots\right) \\
&= \frac{4}{3\sqrt{\pi}}\left(\nu^{3/2} + \frac{\pi^2}{8}\nu^{-1/2} + \ldots\right) \\
&= \frac{4}{3\sqrt{\pi}}\left((\beta\mu')^{3/2} + \frac{\pi^2}{8}(\beta\mu')^{-1/2} + \ldots\right) \qquad (7.174) \\
&= \frac{\langle n\rangle}{g}\left(\frac{2\pi\hbar^2}{m}\beta\right)^{3/2}
\end{aligned}$$

which, for $T \rightarrow 0$, reduces to

$$\frac{4}{3\sqrt{\pi}}\epsilon_F^{3/2} = \frac{\langle n\rangle}{g}\left(\frac{2\pi\hbar^2}{m}\right)^{3/2}$$

$$\rightarrow \quad \epsilon_F = \frac{2\pi\hbar^2}{m}\left(\frac{3\sqrt{\pi}\,\langle n\rangle}{4g}\right)^{2/3}$$

$$= \frac{\hbar^2}{2m}\left(\frac{6\pi^2\langle n\rangle}{g}\right)^{2/3} \tag{7.175}$$

as promised in (7.169f).

(7.174) can now be written as

$$(\beta\epsilon_F)^{3/2} = (\beta\mu')^{3/2} + \frac{\pi^2}{8}(\beta\mu')^{-1/2} + \ldots$$

$$\rightarrow \quad (\beta\mu')^{3/2} = (\beta\epsilon_F)^{3/2} - \frac{\pi^2}{8}(\beta\mu')^{-1/2} + \ldots$$

$$= (\beta\epsilon_F)^{3/2}\left[1 - \frac{\pi^2}{8}\frac{(\beta\mu')^{-1/2}}{(\beta\epsilon_F)^{3/2}} + \ldots\right]$$

$$\therefore \quad \mu' = \epsilon_F\left[1 - \frac{\pi^2}{8\beta^2}\frac{1}{(\mu')^{1/2}\epsilon_F^{3/2}} + \ldots\right]^{2/3}$$

$$\approx \epsilon_F\left[1 - \frac{\pi^2}{12}\left(\frac{k_B T}{\epsilon_F}\right)^2 + \ldots\right] \qquad \text{for } \mu' \approx \epsilon_F \tag{7.176}$$

For arbitrary temperatures, μ' is obtained by solving numerically the equation [see (7.169c)]

$$\frac{\langle n\rangle}{g}\left(\frac{2\pi\hbar^2}{m}\beta\right)^{3/2} = f_{3/2}\left(e^{\beta\mu'}\right) \qquad [\,\langle n\rangle = \text{const.}\,] \tag{7.176a}$$

See Fig.7.21 for a plot of the result.

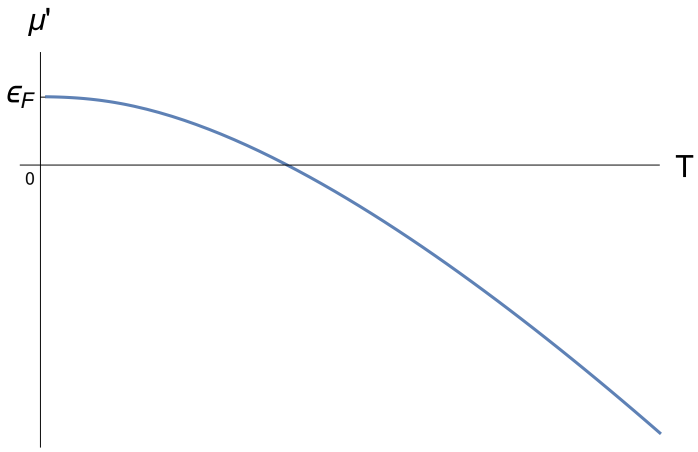

Fig.7.21. Temperature dependence of the chemical potential for a Fermi gas.

Similarly, the internal energy is

$$U = \langle H \rangle = \sum_l \langle n_l \rangle \epsilon_l$$

$$= \frac{4\pi V}{(2\pi)^3} \int_0^\infty dk\, k^2 \frac{\hbar^2 k^2}{2m} \frac{g}{\frac{1}{z} e^{\beta \hbar^2 k^2/2m} + 1}$$

$$= \frac{4\pi V}{(2\pi)^3} \frac{\hbar^2}{2m} \left(\frac{4\pi}{\lambda_T^2}\right)^{5/2} \frac{1}{2} \int_0^\infty dx\, x^{3/2} \frac{g}{\frac{1}{z} e^x + 1} \qquad x = \beta \frac{\hbar^2 k^2}{2m} = \frac{\lambda_T^2}{4\pi} k^2$$

$$= 8\sqrt{\pi}\, V \frac{\hbar^2 g}{2m\lambda_T^5} \Gamma\left(\frac{5}{2}\right) F_{3/2}(\beta\mu') \qquad \text{[(7.168b) used.]}$$

$$= 8\sqrt{\pi}\, V \frac{\hbar^2 g}{2m\lambda_T^5} \Gamma\left(\frac{5}{2}\right) f_{5/2}(z) \qquad \text{[(7.168d) used.]}$$

$$= 6\pi V \frac{\hbar^2 g}{2m\lambda_T^5} f_{5/2}(z) \qquad (7.176b)$$

Following what was done for $f_{3/2}(z)$, we start with

$$f_{5/2}(z) = \frac{4}{3\sqrt{\pi}} \int_0^\infty dy\, y^{3/2} \frac{1}{e^{y-\nu} + 1} \qquad (7.176c)$$

Using

$$d \frac{y^{5/2}}{e^{y-\nu} + 1} = \left[\frac{5}{2} \frac{y^{3/2}}{e^{y-\nu} + 1} - \frac{y^{5/2} e^{y-\nu}}{(e^{y-\nu} + 1)^2}\right] dy$$

we can integrate by part (7.176c) to get

$$f_{5/2}(z) = \frac{8}{15\sqrt{\pi}} \left[\frac{y^{5/2}}{e^{y-\nu} + 1} \Bigg|_0^\infty + \int_0^\infty dy \frac{y^{5/2} e^{y-\nu}}{(e^{y-\nu} + 1)^2} \right]$$

$$= \frac{8}{15\sqrt{\pi}} \int_0^\infty dy \frac{y^{5/2} e^{y-\nu}}{(e^{y-\nu} + 1)^2}$$

$$= \frac{8}{15\sqrt{\pi}} \int_{-\nu}^\infty dt \frac{(t+\nu)^{5/2} e^t}{(e^t + 1)^2}$$

$$= \frac{8}{15\sqrt{\pi}} \int_{-\nu}^\infty dt \frac{\nu^{5/2} e^t}{(e^t + 1)^2} \left(1 + \frac{t}{\nu}\right)^{5/2}$$

$$= \frac{8}{15\sqrt{\pi}} \int_{-\nu}^{\infty} dt \frac{\nu^{5/2} e^t}{(e^t+1)^2} \left[1 + \frac{5}{2}\frac{t}{\nu} + \frac{15}{8}\left(\frac{t}{\nu}\right)^2 + \ldots \right]$$

For low temperatures,

$$f_{5/2}(z) \approx \frac{8}{15\sqrt{\pi}} \int_{-\infty}^{\infty} dt \frac{\nu^{5/2} e^t}{(e^t+1)^2} \left[1 + \frac{5}{2}\frac{t}{\nu} + \frac{15}{8}\left(\frac{t}{\nu}\right)^2 + \ldots \right]$$

$$= \frac{8}{15\sqrt{\pi}} \nu^{5/2} \left(I_0 + \frac{5}{2\nu} I_1 + \frac{15}{8\nu^2} I_2 + \ldots \right)$$

$$= \frac{8}{15\sqrt{\pi}} \nu^{5/2} \left(1 + \frac{5}{8\nu^2} \pi^2 + \ldots \right)$$

so that (7.176b) becomes

$$U = \frac{16}{5} \sqrt{\pi}\, V \frac{\hbar^2 g}{2 m \lambda_T^5} \nu^{5/2} \left(1 + \frac{5\pi^2}{8\nu^2} + \ldots \right)$$

Using

$$\frac{\hbar^2 g}{2 m \lambda_T^5} \nu^{5/2} = \frac{\hbar^2 \langle n \rangle}{2 m \lambda_T^2} \frac{\nu^{5/2}}{f_{3/2}(z)} \qquad \text{[(7.167) used.]}$$

$$= \frac{\hbar^2 \langle n \rangle}{2 m \lambda_T^2} \nu \frac{3\sqrt{\pi}}{4} \left(1 + \frac{\pi^2}{8} \nu^{-2} \ldots \right)^{-1} \qquad \text{[(7.172) used.]}$$

$$= \frac{\langle n \rangle}{4\pi} \mu' \frac{3\sqrt{\pi}}{4} \left(1 + \frac{\pi^2}{8} \nu^{-2} \ldots \right)^{-1} \qquad \text{[(7.135) used.]}$$

we have

$$U = \frac{16}{5} \sqrt{\pi}\, V \frac{\langle n \rangle}{4\pi} \mu' \frac{3\sqrt{\pi}}{4} \left(1 + \frac{\pi^2}{8} \nu^{-2} \ldots \right)^{-1} \left(1 + \frac{5\pi^2}{8\nu^2} + \ldots \right)$$

$$= \frac{3}{5} \langle N \rangle \mu' \left(1 - \frac{\pi^2}{8} \nu^{-2} \ldots \right) \left(1 + \frac{5\pi^2}{8\nu^2} + \ldots \right)$$

$$= \frac{3}{5} \langle N \rangle \mu' \left[1 + \frac{\pi^2}{2} \left(\frac{k_B T}{\mu'} \right)^2 + \ldots \right]$$

$$\approx \frac{3}{5} \langle N \rangle \epsilon_F \left[1 - \frac{\pi^2}{12} \left(\frac{k_B T}{\epsilon_F} \right)^2 + \ldots \right] \left[1 + \frac{\pi^2}{2} \left(\frac{k_B T}{\epsilon_F} \right)^2 + \ldots \right] \qquad \text{[(7.176) used.]}$$

$$\approx \frac{3}{5} \langle N \rangle \epsilon_F \left[1 + \frac{5\pi^2}{12} \left(\frac{k_B T}{\epsilon_F} \right)^2 + \ldots \right] \qquad (7.177)$$

The heat capacity at constant volume is therefore

$$C_V = \left(\frac{\partial U}{\partial T} \right)_{V,\langle N \rangle} = \frac{\pi^2}{2} \langle N \rangle \frac{k_B^2}{\epsilon_F} T + \ldots \qquad (7.178)$$

Ex.7.8.

Compute var(N) for a Fermi gas for $T \approx 0$.

Answer

As usual, the simplest way to calculate var(N) is to use

$$\text{var}(N) = k_B T \left(\frac{\partial \langle N \rangle}{\partial \mu'} \right)_{T,V} \qquad (1)$$

$T \approx 0$, (7.174) gives

$$\frac{\langle N \rangle \lambda_T^3}{V g} = \frac{4}{3\sqrt{\pi}} \left((\beta\mu')^{3/2} + \frac{\pi^2}{8} (\beta\mu')^{-1/2} + \ldots \right) \qquad (2)$$

so that

$$\operatorname{var}(N) = k_B T \frac{V g}{\lambda_T^3} \frac{4}{3\sqrt{\pi}} \left(\frac{3}{2} (\mu')^{1/2} \beta^{3/2} + \ldots \right)$$

$$= k_B T V g \left(\frac{m k_B T}{2 \pi \hbar^2} \right)^{3/2} \frac{4}{3\sqrt{\pi}} \left(\frac{3}{2} \frac{(\mu')^{1/2}}{(k_B T)^{3/2}} + \ldots \right)$$

$$\approx \frac{2}{\sqrt{\pi}} g V \left(\frac{m}{2 \pi \hbar^2} \right)^{3/2} k_B T (\mu')^{1/2}$$

$$= \frac{2}{\sqrt{\pi}} g V \left(\frac{m}{2 \pi \hbar^2} \right)^{3/2} k_B T \left(\frac{\hbar^2}{2 m} \right)^{1/2} \left(\frac{6 \pi^2 \langle n \rangle}{g} \right)^{1/3} \qquad \text{[(7.175) used.]}$$

$$= \frac{1}{2 \pi^2} g V \frac{m}{\hbar^2} k_B T \left(\frac{6 \pi^2 \langle n \rangle}{g} \right)^{1/3}$$

$$= V \frac{m}{\hbar^2} k_B T \left(\frac{3 \pi^2 g^2 \langle N \rangle}{4 \pi^4 V} \right)^{1/3} \qquad (6)$$

$$\to \quad \operatorname{var}\left(\frac{N}{V} \right) = \frac{1}{V} \frac{m}{\hbar^2} k_B T \left(\frac{3 \pi^2 g^2 \langle n \rangle}{4 \pi^4} \right)^{1/3}$$

$$\longrightarrow 0 \qquad \text{Thermodynamic limit.}$$

Ex.7.9.

Compute the density of states at the Fermi surface for an ideal Fermi gas confined to a cubic box of volume $V = L^3$.

Answer

Imposing the periodic boundary conditions, we have

$$\boldsymbol{p} = \hbar \boldsymbol{k} = \hbar \frac{2\pi}{L} \boldsymbol{l} = \hbar \frac{2\pi}{L} (l_x, l_y, l_z) \qquad l_j = 0, \pm 1, \pm 2, \ \ldots. \qquad (1)$$

For each spin degree of freedom, the density of allowable points in k-space is therefore

$$\rho(\boldsymbol{k}) = \left(\frac{L}{2\pi} \right)^3 = \frac{V}{(2\pi)^3}$$

The density of states $\rho(\epsilon)$ is defined as

$$\int \rho(\boldsymbol{k})\, d^3 k = \int \rho(\epsilon)\, d\epsilon = N \qquad \text{[For each spin degree of freedom.]}$$

Now

$$\epsilon = \frac{\hbar^2 k^2}{2 m} \quad \to \quad d\epsilon = \frac{\hbar^2 k}{m} dk = \frac{\hbar}{m} \sqrt{2 m \epsilon}\, dk$$

$$\therefore \quad \frac{V}{(2\pi)^3} 4 \pi k^2\, dk = \rho(\epsilon)\, d\epsilon$$

$$= \frac{V}{(2\pi)^3} 4 \pi \frac{2 m \epsilon}{\hbar^2} \frac{m}{\hbar \sqrt{2 m \epsilon}} d\epsilon$$

$$= \frac{V}{\pi^2} \frac{m^{3/2}}{\sqrt{2}\, \hbar^3} \sqrt{\epsilon}\, d\epsilon \qquad (3)$$

$$\to \quad \rho(\epsilon) = \frac{V}{\pi^2} \frac{m^{3/2}}{\sqrt{2}\, \hbar^3} \sqrt{\epsilon} \qquad (3a)$$

At the Fermi surface, we have

$$\rho(\epsilon_F) = \frac{V}{\pi^2}\frac{m^{3/2}}{\sqrt{2}\,\hbar^3}\sqrt{\epsilon_F} \tag{4}$$

Code

```
In[1]:= nL[ϵ_, μ_, T_] := 1/(ⅇ^((ϵ-μ)/T) + 1)

In[4]:= (* Fig.7.16 *)
Plot[{nL[ϵ, 1, 0.001], nL[ϵ, .9, .1]}, {ϵ, 0, 1.5},
          PlotRange → {{-.1, 1.5}, {-.1, 1.1}},
          AxesLabel → { "ϵ", "⟨n_ℓ⟩" },

 Ticks → {{0, {1, "     μ′(0) "}, {.9, "μ′(T)      "}}, {{.5, "g/2"}, {1, "g"}}},
          PlotLegends → {"T = 0", "T > 0"},
          Epilog → {Text["0", -.03 {1, 1.5}],
                    Dashed, Line[{{0, .5}, {1, .5}}],
                            Line[{{.9, 0}, {.9, .5}}]
                    }
]

In[∘]:= (* Fig.7.17 *)
Plot[{-PolyLog[3/2, -z], -PolyLog[5/2, -z]}, {z, 0, 50},
          PlotRange → {{-1, 55}, {-.1, 15}},
          AxesLabel → { "z", "f_j" },
          PlotLegends → {"f_3/2", "f_5/2"}
]

In[∘]:= zvB[v_, vc_] := { z /. FindRoot[PolyLog[3/2, z] v == vc Zeta[3/2], {z, v}] // Chop   v > vc
                          1                                                               v ≤ vc

In[∘]:= zvF[v_, T_] := z /. FindRoot[-PolyLog[3/2, -z] v T^(3/2) == 1, {z, v}] // Chop

In[∘]:= PB[v_, T_] := (vc = 1/(T^(3/2) Zeta[3/2]); T^(5/2) PolyLog[5/2, zvB[v, vc]])

In[∘]:= PF[v_, T_] := (z = zvF[v, T]; T/v PolyLog[5/2, -z]/PolyLog[3/2, -z] // Chop)

In[∘]:= (* Fig.7.18 *)
T = 1;
Plot[{PB[v, T], PF[v, T], T/v}, {v, .01, 3},
          PlotRange → {{-.2, 3.2}, {-.2, 3}},
          AxesOrigin → {0, 0},
          Ticks → None,
          AxesLabel → { "v", "P" },
          PlotLegends → {"BE", "FD", "Cl"},
          Epilog → Text["0", -.1 {1, 1.5}]
]
```

In[•]:=
```
(* Fig.7.19 *)
Plot[{PolyLog[3/2, ⅇ^μ], -PolyLog[3/2, -ⅇ^μ]}, {μ, -5, 5},
        PlotRange → {{-5, 5}, {-.4, 4}},
        AxesLabel → { "βμ' ", "⟨n⟩λT^3" },
        Ticks → {None, {{PolyLog[3/2, 1], "ζ(3/2)"}}},
        PlotLegends → {"BE", "FD"},
        Epilog → Text["0", -.1 {1, 1.5}]
]

(* Fig.7.20 *)
ym = 15; v = 3;

Plot[ⅇ^(y-v)/(ⅇ^(y-v) + 1)^2, {y, -ym + v, ym},
        PlotRange → {{-ym + v, ym}, {-.03, .3}},
        AxesLabel → { "y", "Δ(y,v) " },
        Ticks → {{{v, "v"}}, {0.25}},
        Epilog → {Text["0", -.05 {ym, .3}],
                Dashed, Line[{{v, 0}, {v, .25}}],
                    Line[{{0, .25}, {v, .25}}]
                }
]
```

In[•]:=
```
(* half-width *)
yh = y /. Solve[ⅇ^(y-v)/(ⅇ^(y-v) + 1)^2 == 1/8, y] /. C[1] → 1;
yh[[1]] - yh[[2]]
```

Out[•]= $-\text{Log}\left[3-2\sqrt{2}\right]+\text{Log}\left[3+2\sqrt{2}\right]$

In[•]:=
```
μs[T_] := T Log[z] /. FindRoot[-PolyLog[3/2, -z] == T^(-3/2), {z, T}] // Chop
```

In[•]:=
```
(* Fig.7.21 *)
Plot[μs[T], {T, 0.03, 3},
      PlotRange → {{-.1, 3}, {-5, 2}},
        AxesLabel → { "T", "μ' " },
        Ticks → {None, {{μs[0.1], "ϵF"}}},
        Epilog → Text["0", -.05 {1, 5}]]
```