8.C. Scaling

As described in §3.H, derivatives of thermodynamic functions, such as the response function (or generalized susceptibility) χ, behave like [c.f. (3.86)]

$$\chi \propto (-\epsilon)^{-\alpha} \qquad\qquad \epsilon = \frac{T - T_C}{T_C}$$

as the system approaches the critical point. α is called the critical exponent. This means, near the critical point, all thermodynamic functions take the form [see §3.H.1]

$$f(\epsilon) = A\,\epsilon^{\lambda}\,(1 + B\,\epsilon^{y} + \ldots) \qquad (3.81)$$

As pointed out by Widom, this can be explained mathematically if the singular part f_s of the thermodynamic function, and hence its derivatives, is a homogenous function [see §8.C.1-2]. In short, as $\epsilon \to 0$, the singular (and hence dominant) part of the system scales with ϵ, i.e., the dependence of f_s on ϵ can be described as a change of the units of measurements on the dynamic variables.

The physical explanation is as follows [see §8.C.3 & §8.D]. As $\epsilon \to 0$, the correlation range $\xi(\epsilon)$ of fluctuations increases indefinitely. Microscopic details such as lattice spacings thus become irrelevant and the characteristic length of the system becomes $\xi(\epsilon)$. Thus, f_s scales with $\xi(\epsilon)$, and hence with ϵ.

8.C.1. Homogeneous Functions

A function $F(x)$ is homogeneous if

$$F(\lambda x) = g(\lambda)\,F(x) \qquad \forall\ \lambda \qquad (8.24)$$

Since

$$\begin{aligned} F(\lambda\,\mu\,x) &= F[\,(\lambda\,\mu)\,x\,] = g(\lambda\,\mu)\,F(x) \\ &= F[\,\lambda\,(\mu\,x)\,] = g(\lambda)\,F(\mu\,x) = g(\lambda)\,g(\mu)\,F(x) \end{aligned} \qquad (8.25)$$

we have

$$g(\lambda\,\mu) = g(\lambda)\,g(\mu) \qquad (8.26)$$

With

$$g'(\lambda) \equiv \frac{d\,g(\lambda)}{d\,\lambda}$$

(8.26) gives

$$\begin{aligned} \frac{\partial g(\lambda\,\mu)}{\partial\,\mu} &= \frac{\partial(\lambda\,\mu)}{\partial\,\mu}\,g'(\lambda\,\mu) = \lambda\,g'(\lambda\,\mu) \\ &= \frac{\partial\,g(\lambda)\,g(\mu)}{\partial\,\mu} = g(\lambda)\,g'(\mu) \end{aligned} \qquad (8.27)$$

Setting $\mu = 1$ and $p = g'(1)$, (8.27) gives

$$\lambda\,g'(\lambda) = g(\lambda)\,g'(1) = g(\lambda)\,p \qquad (8.28)$$

$$\to \qquad \frac{d\,g(\lambda)}{g(\lambda)} = p\,\frac{d\,\lambda}{\lambda}$$

$$\therefore \qquad \ln g(\lambda) = p \ln \lambda + C$$

$$g(\lambda) = \lambda^{p}\,e^{C} \qquad (8.29a)$$

Setting $\lambda = 1$ in (8.26) gives

$$g(\mu) = g(1)\,g(\mu) \to \qquad g(1) = 1$$

Doing the same to (8.29a) gives

$$g(1) = e^C$$

so that (8.29a) becomes

$$g(\lambda) = \lambda^p \qquad (8.29)$$

(8.24) then simplifies to

$$F(\lambda x) = \lambda^p F(x) \qquad (8.30)$$

whereupon $F(x)$ is said to be homogeneous to degree p.

Setting $\lambda = x^{-1}$, (8.30) gives

$$F(1) = x^{-p} F(x) \;\rightarrow\; F(x) = F(1)\, x^p \qquad (8.31)$$

Thus, a homogeneous function of a single argument is just a monomial (a polynomial of a single term) with possibly non-integer and negative powers.

Note that the transformation

$$F(x) \rightarrow F(\lambda x)$$

does not change the functional form of F. It merely changes the scale (measuring unit) of x. Thus, it is often referred to as scaling.

Consider now a homogeneous function of two or more arguments $x, y, z, \ldots$. (8.30) is generalized to

$$F(\mu^a x, \mu^b y, \mu^c z, \ldots) = \mu^m F(x, y, z, \ldots) \qquad (8.32a)$$

or simply

$$F(\lambda^p x, \lambda^q y, \lambda^r z, \ldots) = \lambda F(x, y, z, \ldots) \qquad (8.32)$$

where

$$\lambda = \mu^m \qquad p = \frac{a}{m} \qquad q = \frac{b}{m} \qquad r = \frac{c}{m} \qquad \ldots$$

Setting

$$\lambda = x^{-1/p}$$

turns (8.32) into

$$F(1, x^{-q/p} y, x^{-r/p} z, \ldots) = x^{-1/p} F(x, y, z, \ldots)$$

or

$$F(x, y, z, \ldots) = x^{1/p} F\left(1, \frac{y}{x^{q/p}}, \frac{z}{x^{r/p}}, \ldots\right) \qquad (8.33)$$

where $F\left(1, \frac{y}{x^{q/p}}, \frac{z}{x^{r/p}}, \ldots\right)$ is invariant under the scaling

$$x \rightarrow \lambda^p x, \qquad y \rightarrow \lambda^q y, \qquad z \rightarrow \lambda^r z, \qquad \ldots \qquad (8.33a)$$

Let all exponents $p, q, r, \ldots$ be non-negative, then, as $x \rightarrow 0$,

$$F\left(1, \frac{y}{x^{q/p}}, \frac{z}{x^{r/p}}, \ldots\right) \approx c_y \left(\frac{y}{x^{q/p}}\right)^{n_y} + c_z \left(\frac{z}{x^{r/p}}\right)^{n_z} + \ldots \qquad (8.33b)$$

where $c_y, c_z, \ldots$ are constants and $n_y, n_z, \ldots$ are non-negative integers such that each term on the R.H.S. represents the most divergent contribution by the given variable.

The resemblance between (8.33-b) and (3.81) is the basis for the Widom scaling.

8.C.2. Widom Scaling

The Gibbs free energy of a magnetic system can be written as

$$G(T, \boldsymbol{B}) = G_r(T, \boldsymbol{B}) + G_s(\epsilon, \boldsymbol{B}) \qquad (8.34)$$

where G_r and G_s denote the regular and singular parts of G, respectively. The "distance" from the critical point is given by

$$\epsilon = \frac{T - T_C}{T_C} \tag{8.34a}$$

where T_C is the critical temperature.

According to Widom, critical exponents of the phase transition can be obtained by assuming G_s to be a homogeneous function:

$$G_s(\lambda^p \epsilon, \lambda^q \boldsymbol{B}) = \lambda\, G_s(\epsilon, \boldsymbol{B}) \tag{8.35}$$

Consider the case of a constant field $\boldsymbol{B} = B\,\hat{\boldsymbol{z}}$. From (8.3), the z-component of the magnetization is given by [see (2.110) with B as the intensive variable]

$$M = -\frac{\partial G}{\partial B}$$

and (8.35) gives

$$-\frac{\partial}{\partial B} G_s(\lambda^p \epsilon, \lambda^q B) = -\frac{\partial (\lambda^q B)}{\partial B} \frac{\partial G_s(\lambda^p \epsilon, \lambda^q B)}{\partial (\lambda^q B)}$$

$$= \lambda^q M(\lambda^p \epsilon, \lambda^q B) \tag{8.37a}$$

$$= -\lambda \frac{\partial G_s(\epsilon, \boldsymbol{B})}{\partial B} = \lambda\, M(\epsilon, B) \tag{8.37}$$

Setting

$$\lambda = (-\epsilon)^{-1/p} \qquad (T < T_C)$$

(8.37a) & (8.37) give

$$(-\epsilon)^{-q/p} M\left[-1, (-\epsilon)^{-q/p} B\right] = (-\epsilon)^{-1/p} M(\epsilon, B)$$

which, for $B = 0$, becomes

$$(-\epsilon)^{-q/p} M(-1, 0) = (-\epsilon)^{-1/p} M(\epsilon, 0)$$

or

$$M(\epsilon, 0) = (-\epsilon)^{(1-q)/p} M(-1, 0) \tag{8.38}$$

The critical exponent β, which gives the degree of the co-existence curve, is defined as [c.f. (3.85)]

$$\Psi \propto (-\epsilon)^{\beta}$$

where Ψ is the order parameter,

In the present case,

$$M(\epsilon, 0) \propto (-\epsilon)^{\beta} \tag{8.36}$$

Comparing it with (8.38) gives

$$\beta = \frac{1-q}{p} \tag{8.39}$$

The critical exponent δ, which gives the degree of the critical isotherm [see §3.H.2], is defined as [c.f. (3.84)]

$$Y \propto \Psi^{\delta} \qquad \text{at } \epsilon = 0$$

where Y is the field (intensive) variable. For our magnetic system, $Y = B$ and $\Psi = | M(\epsilon, B) |$. Absolute value is used here because $(-)^{\delta}$ is a complex number if δ is not an integer. If the system is paramagnetic, B and M have the same sign so that

$$B \propto | M(0, B) |^{\delta} \operatorname{sign} M$$

$$\rightarrow \qquad M(0, B) \propto | B |^{1/\delta} \operatorname{sign} B \tag{8.40}$$

Setting $\epsilon = 0$ in (8.37) gives

$$\lambda^q M(0, \lambda^q B) = \lambda\, M(0, B)$$

For $\lambda = B^{-1/q}$, this becomes

$$B^{-1} M(0, 1) = B^{-1/q} M(0, B)$$

or

$$M(0, B) = B^{(1-q)/q} M(0, 1) \qquad (8.41)$$

Comparing with (8.40) gives

$$\delta = \frac{q}{1-q} \qquad (8.42)$$

Consider now the magnetic susceptibility and the accompanying critical exponents

$$\chi = \left(\frac{\partial M}{\partial B}\right)_T = -\left(\frac{\partial^2 g}{\partial B^2}\right)_T$$

$$\propto \begin{cases} (-\epsilon)^{-\gamma'} & T < T_C \\ \epsilon^{-\gamma} & T > T_C \end{cases} \qquad (8.43)$$

Differentiating (8.37-a) gives

$$\lambda^q \frac{\partial M(\lambda^p \epsilon, \lambda^q B)}{\partial B} = \lambda^{2q} \chi(\lambda^p \epsilon, \lambda^q B) \qquad (8.44a)$$

$$= \lambda \frac{\partial M(\epsilon, B)}{\partial B} = \lambda \chi(\epsilon, B) \qquad (8.44)$$

For the static case, $B = 0$, we have

$$\lambda^{2q} \chi(\lambda^p \epsilon, 0) = \lambda \chi(\epsilon, 0)$$

Setting $\lambda = \epsilon^{-1/p}$ gives

$$\epsilon^{-2q/p} \chi(1, 0) = \epsilon^{-1/p} \chi(\epsilon, 0)$$

or

$$\chi(\epsilon, 0) = \epsilon^{(1-2q)/p} \chi(1, 0) \qquad (8.45)$$

Comparing with (8.43) gives

$$\gamma = \frac{2q-1}{p} \qquad (8.46)$$

Setting $\epsilon \to -\epsilon$ in (8.45) gives

$$\gamma' = \frac{2q-1}{p} = \gamma \qquad (8.46a)$$

The heat capacity at constant B and its accompanying exponent are given by [cf.(3.86)]

$$C_B = -T\left(\frac{\partial^2 g}{\partial T^2}\right)_B \propto \begin{cases} (-\epsilon)^{-\alpha'} & T < T_C \\ \epsilon^{-\alpha} & T > T_C \end{cases} \qquad (8.48)$$

From (8.34a), we have

$$T = T_C (1+\epsilon) \qquad \frac{d\epsilon}{dT} = \frac{1}{T_C} \qquad \frac{\partial}{\partial T} = \frac{1}{T_C} \frac{\partial}{\partial \epsilon}$$

so that (8.48) becomes

$$C_B = -\frac{1+\epsilon}{T_C}\left(\frac{\partial^2 g}{\partial \epsilon^2}\right)_B$$

Differentiating (8.35) twice gives

$$\lambda^{2p} C_B (\lambda^p \epsilon, \lambda^q B) = \lambda C_B (\epsilon, B) \qquad (8.48)$$

For the static case,

$$\lambda^{2p} C_B(\lambda^p \epsilon, 0) = \lambda C_B(\epsilon, 0)$$

Setting $\lambda = \epsilon^{-1/p}$ then gives

$$\epsilon^{-2}\, C_B(1,0) = \epsilon^{-1/p}\, C_B(\epsilon, 0)$$

or

$$C_B(\epsilon, 0) = \epsilon^{(1-2p)/p}\, C_B(1,0) \tag{8.49}$$

Comparing with (8.48) gives

$$\alpha = 2 - \frac{1}{p} \tag{8.50}$$

Setting $\epsilon \to -\epsilon$ in (8.49) gives

$$\alpha' = 2 - \frac{1}{p} = \alpha \tag{8.50a}$$

(8.39), (8.42), (8.46) and (8.50) give us 4 (or 6 if we include the primed ones) exponents in terms of two parameters p & q. This means these exponents are not independent.

(8.42) can be inverted to give

$$q = \frac{\delta}{1+\delta} \tag{8.54}$$

which can be put into (8.39) to give

$$p = \frac{1}{\beta(1+\delta)} \tag{8.53}$$

(8.46) then becomes

$$\gamma = \gamma' = \beta(\delta - 1) \tag{8.51}$$

while (8.50) gives

$$\alpha + \beta(1+\delta) = 2 \tag{8.52}$$

The experimental values of the exponents are [see Table 8.1 of Reichl's text]

$$\alpha = 0 \sim 0.2, \qquad \beta = 0.3 \sim 0.4, \qquad \gamma = 1.2 \sim 1.4, \qquad \delta = 4 \sim 5$$

giving

$$\beta(\delta - 1) = 0.9 \sim 1.6$$
$$\beta(1+\delta) = 1.5 \sim 2.4$$
$$2 - \alpha = 1.8 \sim 2$$

Thus, (8.51-2) agree with the measured values.

Now, the exponents given by the mean field theory, namely,

$$\alpha = 0, \quad \beta = \frac{1}{2}, \quad \delta = 3, \quad \gamma = 1$$

mostly fall outside the range of measured values. However, it is easy to verify that they satisfy (8.51-2) exactly.

This may be interpreted as saying that scaling is so essential to the critical phenomena that even the crudest of theoretical model (e.g., mean field theory) satisfies its general conclusions such as (8.51-2).

8.C.3. Kadanoff Scaling

Ref. L.P.Kadanoff, Physics 2, 263 (1966).

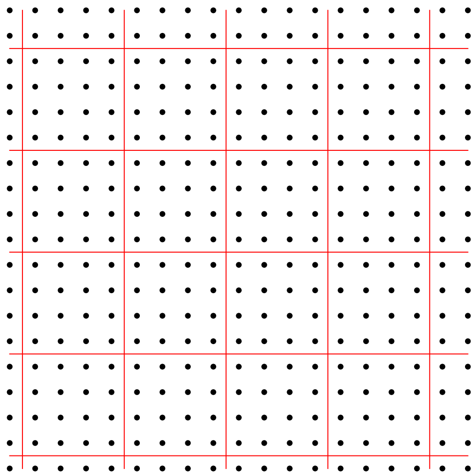
Consider the d-D Ising model with nearest-neighbor interaction. The Hamiltonian is

$$H\{S\} = -K \sum_{(i,j)}^{\Gamma N/2} S_i S_j - B \sum_{i=1}^{N} S_i \tag{8.55}$$

where K is the spin-spin interaction energy, B the applied magnetic induction, N the number of sites, Γ the number of nearest neighbors at each site and the 1st sum is over all $\frac{1}{2}\Gamma N$ pairs of nearest neighbors.

Comment: Since all spins are assumed to be along the z-direction, the Ising model is inherently layer-like. (8.55) is therefore fine for $d \le 2$. However, for $d = 3$, interactions between neighboring spins in the same x-y plane are obviously different from those between neighboring spins on different x-y planes.

Let a be the lattice constant. The scaling operation is carried out by dividing the (cubic) lattice into blocks of side length $L\,a$ so that each block contains L^d sites (or spins). The number of blocks is therefore $\frac{N}{L^d}$.



Blocking (red lines) of square lattice with $L = 4$. Lattice sites are shown as black dots.

The total spin in block I is

$$S_I' = \sum_{i \in I} S_i \qquad (8.56)$$

Since $S_i = \pm 1$ and there are L_d spins in each block, we have

$$-L_d \le S_I' \le L_d$$

It is useful to define

$$S_I' = Z\, S_I \qquad (8.57)$$

where

$$S_I = \pm 1 \qquad 0 \le Z \le L^d$$

If we choose $L\,a << \xi$, where ξ is the correlation length of spin fluctuation, then the spins within each block will be highly correlated. Therefore, $Z \lesssim L^d$ so that we can write

$$Z = L^y \qquad \text{with} \qquad y \lesssim d \qquad (8.57a)$$

Since only nearest neighbor spins interact, likewise the spin blocks. The block Hamiltonian thus takes the form

$$H\{S_L\} = -K_L \sum_{(I,J)}^{\Gamma N L^{-d}/2} S_I\, S_J - B_L \sum_{I=1}^{N L^{-d}} S_I \qquad (8.58)$$

where X_L denotes the effective value of X for blocks.

The block field B_L can be calculated easily by carrying out the blocking operation. Thus, the field term in (8.55) gives

$$B \sum_{i=1}^{N} S_i = B \sum_{I=1}^{N/L^d} \sum_{i \in I} S_i = B \sum_{I=1}^{N/L^d} S_I{}' = B\,Z \sum_{I=1}^{N/L^d} S_I \qquad (8.62)$$

Comparing with (8.58) gives

$$B_L = B\,Z = B\,L^y \qquad (8.63)$$

Note that $H\{S_L\}$ is just $H\{S\}$ with the length scale increased by a factor of L. Near the critical point where the correlation length becomes the characteristic length, we expect its block value (in units of $L\,a$) to shrink by the same factor, i.e.,

$$\xi_L(\epsilon_L, B_L) = \frac{1}{L}\,\xi(\epsilon, B) \qquad (8.60)$$

With $L\,a >> \xi$, each block can attain local equilibrium so that

$$g(\epsilon_L, B_L) = L^d\,g(\epsilon, B) \qquad (8.59)$$

where $g(\epsilon_L, B_L)$ and $g(\epsilon, B)$ are the equilibrium Gibbs free energies per block and per site, respectively.

In the spirit of Widom scaling, all block variables should scale with L. In fact, comparing (8.59) with (8.35) suggests

$$L^d = \lambda \qquad (8.61a)$$

If we set [see also (8.63)]

$$\epsilon_L = L^x\,\epsilon \qquad B_L = L^y\,B \qquad x, y > 0 \quad (8.61)$$

then (8.59) becomes

$$g(L^x\,\epsilon, L^y\,B) = L^d\,g(\epsilon, B) \qquad (8.64)$$

Using (8.61a) with (8.35) gives

$$x = p\,d \qquad y = q\,d \qquad (8.64a)$$

which links the Kadanoff model to the results in §8.C.2. Comparing with (8.57a) gives

$$q \lesssim 1 \qquad (8.65)$$

in agreement with (8.54).

The Kadanoff scaling allows us to introduce two new critical exponents associated with the spatial correlations of spin fluctuations. The block correlation function (of spin fluctuations) is defined as

$$C(r_L, \epsilon_L) = \left\langle \left(S_I - \langle S_I \rangle\right)\left(S_J - \langle S_J \rangle\right)\right\rangle$$
$$= \langle S_I\,S_J \rangle - \langle S_I \rangle \langle S_J \rangle \qquad (8.66)$$

where r is the distance between blocks I & J and r_L is r in units L times larger, i.e.,

$$r_L = \frac{r}{L} \qquad (8.68)$$

Using (8.57), we can write (8.66) as

$$C(r_L, \epsilon_L) = \frac{1}{Z^2}\left(\langle S_I{}'\,S_J{}' \rangle - \langle S_I{}' \rangle \langle S_J{}' \rangle\right)$$
$$= \frac{1}{Z^2} \sum_{i \in I} \sum_{j \in J} \left(\langle S_i\,S_j \rangle - \langle S_i \rangle \langle S_j \rangle\right) \qquad \text{[(8.56) used.]}$$

As $\epsilon \to 0$, the averages $\langle S_i \rangle$ and $\langle S_i\,S_j \rangle$ depend only on which blocks the spins are in. Hence,

$$C(r_L, \epsilon_L) \approx \frac{1}{Z^2}\left(L^d\right)^2 \left(\langle S_i\,S_j \rangle - \langle S_i \rangle \langle S_j \rangle\right) \qquad \text{for } \epsilon \to 0$$
$$= \frac{1}{Z^2}\left(L^d\right)^2 C(r, \epsilon) \qquad (8.67)$$

Using (8.61), (8.63) & (8.68), we can write (8.67) as

$$C\left(\frac{r}{L}, \epsilon L^{x}\right) = L^{2(d-y)}\, C(r, \epsilon) \tag{8.69}$$

Setting $L = \frac{r}{a}$ gives

$$C\left(a, \epsilon\left(\frac{r}{a}\right)^{x}\right) = \left(\frac{r}{a}\right)^{2(d-y)} C(r, \epsilon)$$

or

$$C(r, \epsilon) = \left(\frac{r}{a}\right)^{2(y-d)} C\left(a, \epsilon\left(\frac{r}{a}\right)^{x}\right) \tag{8.70}$$

The critical exponent ν is defined for the correlation length as

$$\xi \propto \epsilon^{-\nu} \text{for } \; T > T_C \tag{8.71}$$

In the mean field approximation [see §8.B.2], we have

$$C(r) \propto \frac{1}{r} e^{-r/\xi} \tag{8.23}$$

with

$$\xi \propto \epsilon^{-1/2} \qquad \rightarrow \qquad \nu = \frac{1}{2}$$

Actually, (8.23) describes the generally behavior of C for $r >> \xi$ and often serves as the definition of ξ. Thus, using (8.71) for $\epsilon \to 0$, the r-dependence of C is expected to be through the combination $\frac{r}{\xi} \propto r\,\epsilon^{\nu}$.

On the other hand, (8.70) shows that this dependence is through $\epsilon\, r^{x}$ or, equivalently, $\epsilon^{1/x}\, r$. Thus,

$$\nu = \frac{1}{x} = \frac{1}{p\,d} \tag{8.72}$$

where (8.64a) was used.

At the critical point $\epsilon = 0$, (8.70) gives

$$C(r, 0) = \left(\frac{r}{a}\right)^{2(y-d)} C(a, 0)$$
$$\propto r^{2(y-d)} \tag{8.73}$$

The critical exponent η is related to the spatial behavior of C at the critical point as

$$C(r, 0) \propto \left(\frac{1}{r}\right)^{d-2+\eta} \tag{8.75}$$
$$= \left(\frac{1}{r}\right)^{1+\eta} \qquad \text{for } d = 3 \tag{8.74}$$

Setting $\xi \to \infty$ in (8.23) gives

$$C(r, 0) \propto \frac{1}{r}$$

so that $\eta = 0$ for the mean field theory.

Comparing (8.75) with (8.73) gives

$$d - 2 + \eta = -2(y - d)$$
$$= -2(q - 1)\,d \qquad \text{[(8.64a) used.]} \tag{8.76}$$

The new exponents can be expressed in terms of the old ones. For example, from (8.50) & (8.72), we have

$$\nu = \frac{2 - \alpha}{d} \tag{8.77}$$

From (8.54) & (8.76), we have

$$\eta = 2 - d\,\frac{\delta - 1}{\delta + 1} = 2 - \frac{d\,\gamma}{2\,\beta + \gamma} \tag{8.78}$$

The Kadanoff scaling thus provides us with 2 new exponents and the accompanying identities between exponents.