8.D. Microscopic Calculation of Critical Exponents

Kadanoff did not carry out the blocking procedure to the extend that can produce actual values for the critical exponents. The task had to await for Wilson and his renormalization group approach.

Consider a system of N spins described by the partition function

$$Z(\boldsymbol{K}, N) = \sum_{\{S_i\}} \exp[-\beta \mathcal{H}(\boldsymbol{K}, \{S_i\}, N)] \tag{8.79}$$

where the effective Hamiltonian takes the form

$$\beta \mathcal{H}(\boldsymbol{K}, \{S_i\}, N) = K_0 + K_1 \sum_i S_i + K_2 \sum_{i,j}^{(1)} S_i S_j + K_3 \sum_{i,j}^{(2)} S_i S_j \tag{8.80}$$

$$+ K_4 \sum_{i,j,k}^{(1)} S_i S_j S_k + \ldots$$

where $\boldsymbol{K} = (K_0, K_1, K_2, \ldots)$ and $\sum^{(\alpha)}$ means summing over only α^{th} nearest neighbors.

For the Ising model [see (8.55)],

$$K_0 = 0 \qquad K_1 = -\beta B \qquad K_2 = -\beta J \qquad K_3 = K_4 = \ldots = 0$$

where J is the spin-spin coupling constant.

We now introduce blocking. Let σ_l be the set of spins in the l^{th} block and the sum

$$S_l = \sum_{S_i \in \sigma_l} S_i$$

then (8.79) can be written as

$$\begin{aligned} Z(\boldsymbol{K}, N) &= \sum_{\{\sigma_l\}} \exp[-\beta \mathcal{H}(\boldsymbol{K}, \{\sigma_l\}, N)] \\ &= \sum_{\{S_l\}} \exp\left[-\beta \mathcal{H}\left(\boldsymbol{K}_L, \{S_l\}, \frac{N}{L^d}\right)\right] \\ &= Z\left(\boldsymbol{K}_L, \frac{N}{L^d}\right) \end{aligned} \tag{8.81}$$

The corresponding Gibbs free energy per (lattice) site is

$$\begin{aligned} g(\boldsymbol{K}) &= \lim_{N\to\infty} \frac{1}{N} \ln Z(\boldsymbol{K}, N) \\ &= \lim_{N\to\infty} \frac{1}{N} \ln Z\left(\boldsymbol{K}_L, \frac{N}{L^d}\right) = \frac{1}{L^d} g(\boldsymbol{K}_L) \end{aligned} \tag{8.82}$$

where $g(\boldsymbol{K}_L)$ is the Gibbs free energy per block. The crucial step is to treat the relation between $\boldsymbol{K}$ & $\boldsymbol{K}_L$ as a (non-linear) transformation

$$\boldsymbol{K}_L = T(\boldsymbol{K}) \tag{8.83}$$

which preserves the functional form of $\mathcal{H}$. Repeated applications of the transformation lead to

$$\boldsymbol{K}_{nL} = T(\boldsymbol{K}_{(n-1)L}) = T^2(\boldsymbol{K}_{(n-2)L}) = \ldots = T^{n-1}(\boldsymbol{K}_L) = T^n(\boldsymbol{K}) \tag{8.84}$$

Now, the easily verified properties

$$T^n T^m = T^m T^n = T^{n+m} \qquad (T^n T^m) T^k = T^n (T^m T^k)$$

make the set $\mathcal{T} = \{T^n; n = 1, 2, \ldots, \infty\}$ an Abelian semi-group and hence the misnomer renormalization group (renormalization is the term used by quantum theorists to describe the re-definition, e.g., $\boldsymbol{K} \to \boldsymbol{K}_L$, of the dynamical parameters). $\mathcal{T}$ is not a group since the inverse of T is not defined. However, what follows makes no use of group theory.

Now, T is implemented by changing some physical parameters that increases the correlation length ξ. At the critical point, ξ is on the order of the size $V^{1/d}$ of the system and the system experiences a qualitative change, i.e., phase transition. This is denoted mathematically as a critical point of T defined by

$$\boldsymbol{K}^* = T(\boldsymbol{K}^*) \qquad \text{at} \quad \epsilon = 0 \tag{8.85}$$

Near the fixed point $\boldsymbol{K}^* = (K_1^*, K_2^*)$, Taylor expansion gives

$$T(\boldsymbol{K}) \approx T(\boldsymbol{K}^*) + (\delta \boldsymbol{K} \cdot \nabla_{\boldsymbol{K}})\, T(\boldsymbol{K}^*) \qquad \delta \boldsymbol{K} = \boldsymbol{K} - \boldsymbol{K}^* \tag{8.85a}$$

or

$$\delta \boldsymbol{K}_L \approx (\delta \boldsymbol{K} \cdot \nabla_{\boldsymbol{K}})\, \boldsymbol{K}^* \qquad \delta \boldsymbol{K}_L = \boldsymbol{K}_L - \boldsymbol{K}^* \tag{8.85b}$$

Setting

$$\partial_{K_j} K_i^* = \mathbb{A}_{ij} \tag{8.85c}$$

(8.85b) becomes

$$\delta K_{L\,i} = \sum_j \delta K_j\, \partial_{K_j} K_i^* = \sum_j \delta K_j\, \mathbb{A}_{ij}$$

or

$$\delta \boldsymbol{K}_L = \mathbb{A} \cdot \delta \boldsymbol{K} \tag{8.86}$$

(8.86) describes the mapping (or motion of a $\boldsymbol{K}$ point) near the fixed point. Its characteristics are determined by the eigenvalues of $\mathbb{A}$. Since $\mathbb{A}$ is in general nonsymmetric, we must distinguish between left and right eigenvectors [see §5.C].

Let $\mathbb{U}$ be the similarity transform that diagonalize $\mathbb{A}$, i.e.,

$$\mathbb{U}\, \mathbb{A}\, \mathbb{U}^{-1} = \boldsymbol{\Lambda} = \text{diag}(\lambda_1, \lambda_2, \ ...) \tag{8.86a}$$

where λ_i are the eigenvalues of $\mathbb{A}$. Then (8.86) gives

$$\mathbb{U} \cdot \delta \boldsymbol{K}_L = \mathbb{U}\, \mathbb{A}\, \mathbb{U}^{-1}\, \mathbb{U} \cdot \delta \boldsymbol{K}$$

or

$$\delta \boldsymbol{u}_L = \boldsymbol{\Lambda} \cdot \delta \boldsymbol{u} \tag{8.88}$$

where

$$\delta \boldsymbol{u}_L = \mathbb{U} \cdot \delta \boldsymbol{K}_L \qquad \delta \boldsymbol{u} = \mathbb{U} \cdot \delta \boldsymbol{K}$$

In components, (8.88) becomes

$$\delta u_{L\,i} = \sum_j \lambda_i\, \delta_{ij}\, \delta u_j \qquad = \lambda_i\, \delta u_i \tag{8.88a}$$

Thus, the i^{th} components of $\delta \boldsymbol{u}$ & $\delta \boldsymbol{u}_L$ describe the mapping along the i^{th} right eigenvector (eigen-curve) of $\mathbb{A}$.

After n successive transformations, (8.88) & (8.88a) become

$$\delta \boldsymbol{u}_{n\,L} = \boldsymbol{\Lambda}^n \cdot \delta \boldsymbol{u}$$
$$\delta u_{n\,L\,i} = (\lambda_i)^n\, \delta u_i \tag{8.88b}$$

Thus, for $\lambda_i > 1$ ($\lambda_i < 1$), each transformation moves the $\boldsymbol{K}$ point away from (towards) the fixed point along the i^{th} eigen-curve.

As an example, consider the case $\boldsymbol{K} = (K_1, K_2)$. Then

$$\mathbb{A} = \begin{pmatrix} \frac{\partial K_1^*}{\partial K_1} & \frac{\partial K_1^*}{\partial K_2} \\ \frac{\partial K_2^*}{\partial K_1} & \frac{\partial K_2^*}{\partial K_2} \end{pmatrix} \tag{8.87}$$

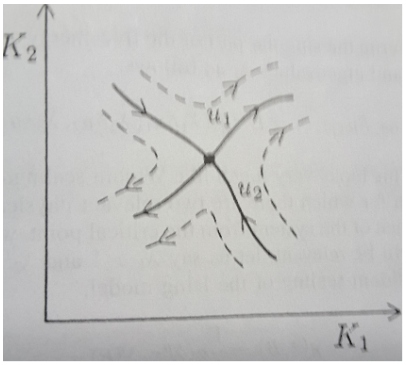
$$\boldsymbol{\Lambda} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{8.89}$$

$$\delta \boldsymbol{u} = \begin{pmatrix} \delta u_1 \\ \delta u_2 \end{pmatrix} \tag{8.90}$$

or

$$\delta u_{n\,L\,1} = (\lambda_1)^n \, \delta u_1 \tag{8.91}$$

$$\delta u_{n\,L\,2} = (\lambda_2)^n \, \delta u_2 \tag{8.92}$$



Hyperbolic fixed point ($\lambda_1 > 1$, $\lambda_2 < 1$).

Note that the eigen-curves are orthogonal to each other so that moving a point along one doesn't not change the point's projections on the others.

If the critical temperature $T_C \neq 0$, then the system can become more orderly as the temperature T goes below T_C. This means ξ can increase beyond $V^{1/d}$ to ∞ as $T \to 0$. Obviously, physical parameters still varies if one increases ξ for $T < T_C$.

Now, a point on eigen-curves with $\lambda_i < 1$ still moves towards the fixed point for transformations (i.e., increasing ξ) at $T < T_C$. Therefore, these eigen-curves cannot represent physical parameters. They are therefore called irrelevant. In contrast, eigen-curves with $\lambda_i > 1$ are called relevant since they can represent physical parameters.

Thus, a fixed point that represent a critical point must be hyperbolic [see figure above] with enough relevant (> 1) eigenvalues to represent the physical parameters present and at least one irrelevant (< 1) eigenvalue to facilitate the approach to phase transition by means of increasing ξ.

Reminder: a point, which represents the state of the system, in the parameter space can also be moved by changing the experimental conditions or thermal fluctuations.

In general, the effect of the blocking on the singular part of the free energy density can be written as

$$g_s(\delta u_1,\, \delta u_2,\, \ldots) = \frac{1}{L^d}\, g_s(\lambda_1\, \delta u_1,\, \lambda_2\, \delta u_2,\, \ldots) \tag{8.93}$$

which is in the form of the Widom scaling [see (8.82)] provided $\lambda_i > 1$.

In the case of the 2-D Ising model, two of the parameters must be relevant. Let these be labeled 1 & 2. Comparing (8.93) with the Widom scaling [see (8.35)]

$$g_s(\epsilon,\, B) = \frac{1}{\lambda}\, g_s(\lambda^p\, \epsilon,\, \lambda^q\, B) \tag{8.94}$$

we get

$$\delta u_1 = \epsilon \qquad \delta u_2 = B$$

and

$$\lambda = L^d \tag{8.95}$$

$$\lambda_1 = \lambda^p = L^{p\,d} \qquad \to \qquad p = \frac{\ln \lambda_1}{d \ln L} \tag{8.96}$$

$$\lambda_2 = \lambda^q = L^{q\,d} \qquad \to \qquad q = \frac{\ln \lambda_2}{d \ln L} \tag{8.97}$$

Exercise 8.1

Compute the critical exponents for 3-spin blocks on the triangular planar lattice for the 2-D nearest neighbor Ising model. Retain terms to lowest order in $\langle V \rangle$.

Graphics

The graph below shows the 2-D hexagonal (triangular) lattice.

Blue lines: original lattice with lattice vectors:

$$\boldsymbol{a} = a\,(1,\,0), \qquad \boldsymbol{b} = a\left(\cos\frac{\pi}{3},\,\sin\frac{\pi}{3}\right) \text{ and auxiliary } \boldsymbol{c} = a\left(-\cos\frac{\pi}{3},\,\sin\frac{\pi}{3}\right).$$

Each lattice cell contains 1 spin.

Green lines: block lattice with with lattice vectors:

$3\,\boldsymbol{a}$, $3\,\boldsymbol{b}$, and auxiliary $3\,\boldsymbol{c}$

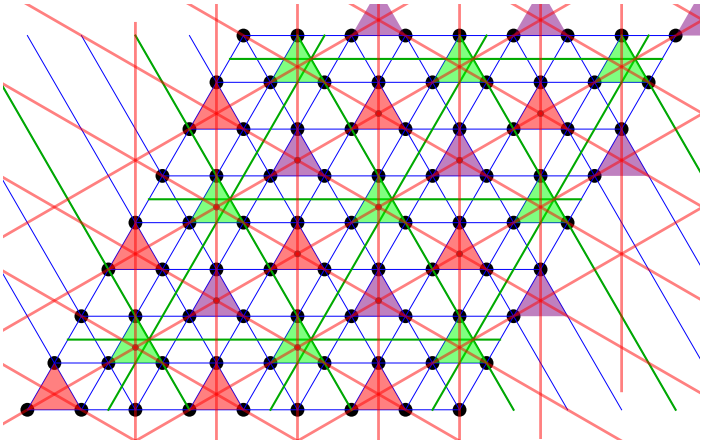
Each lattice unit cell contains three 3-spin blocks shown as triangles of different colors.

Red lines: block lattice with lattice vectors:

$$\boldsymbol{p} = \sqrt{3}\;a\left(\cos\frac{\pi}{6},\,\sin\frac{\pi}{6}\right), \quad \boldsymbol{q} = \sqrt{3}\;a\,(0,\,1) \text{ and auxiliary } \boldsymbol{r} = \sqrt{3}\;a\left(-\cos\frac{\pi}{6},\,\sin\frac{\pi}{6}\right).$$

Each lattice cell contains one 3-spin block. Lattice is original lattice rotated by an angle of $\frac{\pi}{6}$ and expanded by a factor of $\sqrt{3}$.

Mathematica code for the graph is in file "Block.nb".



Answer

The Hamiltonian is [see (8.55)]

$$\beta H = -K \sum_{(i,j)} s_i\, s_j - B \sum_i s_i \qquad s_i = \pm 1 \qquad (1)$$

Note that both $K = \beta J$ and B are dimensionless, which means $B = \beta\,\mu\,B_f$ where μ is the magnetic moment and B_f the magnetic induction.

Consider the blocking shown in the figure above. After increasing the length scale (or lattice constant) from a to $3\,a$ [green lines], the spin “unit” is expanded from a single spin to a 3-spin block marked by a colored equilateral triangle in the figure. The area of the lattice unit cell is increased by a factor of $3^2 = 9$, while the number of spin units in the lattice unit cell is increased from 1 to 3. The

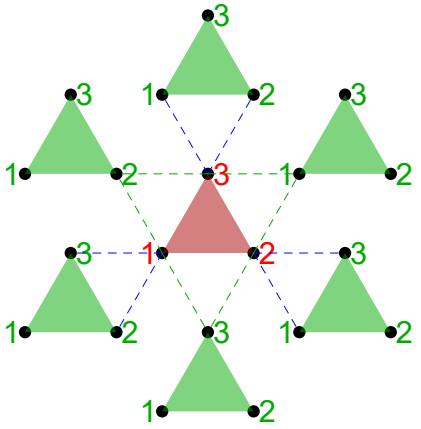
area occupied by each spin unit is therefore increased by a factor of $\frac{9}{3} = 3$. Thus, from the point of view of the spin units, the length scale of the system is increased a factor of $\sqrt{3}$.

A better way to describe the block lattice is to use the lattice vectors

$$\boldsymbol{p} = \sqrt{3}\, a\left(\cos\frac{\pi}{6}, \sin\frac{\pi}{6}\right), \quad \boldsymbol{q} = \sqrt{3}\, a\,(0,\, 1) \text{ and auxiliary } \boldsymbol{r} = \sqrt{3}\, a\left(-\cos\frac{\pi}{6}, \sin\frac{\pi}{6}\right)$$

which describes a lattice [see red lines] obtained from the original one by a rotation of $\frac{\pi}{6}$, accompanied by an increase of the length scale by a factor of $\sqrt{3}$. Each unit cell now contains only one spin unit.

Note that in both the original and block lattice, the number of nearest neighbors for each spin unit is 6, as shown in the figure below.



Each 3-spin block will be characterized by two parameters

$$S_I = \text{sign}\left(\sum_{i\in I} s_i\right) = \text{sign}\left(s_1^I + s_2^I + s_3^I\right)$$

$$\sigma_I = \left|\sum_{i\in I} s_i\right| = \left|\, s_1^I + s_2^I + s_3^I \,\right| \qquad \text{(1a)}$$

so that

$$\sum_{i\in I} s_i = S_I\, \sigma_I \qquad S_I = \pm 1, \qquad \sigma_I = 1,\, 3 \qquad \text{(1b)}$$

where the state $\sigma_I = 1$ is 3-fold degenerate [see Table 1.c].

Each 3-spin block can have 8 spin configurations, as shown in the following table

$s_1\, s_2\, s_3$	↓ ↑ ↑	↑ ↓ ↑	↑ ↑ ↓	↑ ↑ ↑	↑ ↓ ↓	↓ ↑ ↓	↓ ↓ ↑	↓ ↓ ↓
$\tilde{\alpha}$	1	2	3	4	5	6	7	8
$(S_I,\, \alpha_I)$	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(−1, 1)	(−1, 2)	(−1, 3)	(−1, 4)
S_I	1	1	1	1	−1	−1	−1	−1
σ_I	1	1	1	3	1	1	1	3
α_I	1	2	3	4	1	2	3	4

(1c)

The corresponding values for S_I & σ_I are listed in rows 4 & 5, respectively.

There are two ways to label these configurations. The 1st is by the single parameter $\tilde{\alpha} = 1,\ \ldots,\ 8$, as shown in the 2nd row of the table. The 2nd is by a pair of parameters $(S_I,\, \alpha_I)$ with $S_I = \pm 1$ and $\alpha_I = 1,\ \ldots,\ 4$, as shown in the 3rd row of the table. Thus, the I^{th} 3-spin block is characterized by a

spin with direction S_I, magnitude σ_I, and "internal" configuration α_I.

The blocking of H proceeds as follows. First, we split H into two parts

$$H(\boldsymbol{K}, \{S_I, \alpha_I\}) = H_0(\boldsymbol{K}, \{S_I, \alpha_I\}) + V(\boldsymbol{K}, \{S_I, \alpha_I\}) \tag{3}$$

where the non-interacting, or unperturbed, part

$$\beta H_0(\boldsymbol{K}, \{S_I, \alpha_I\}) = -K \sum_I \sum_{i,j \in I} s_i s_j \tag{4}$$

contains all interactions within each block, while the rest are dumped into

$$\beta V(\boldsymbol{K}, \{S_I, \alpha_I\}) = -K \sum_{(I,J)} \sum_{i \in I} \sum_{j \in J} s_i s_j - B \sum_I \sum_{i \in I} s_i \tag{5}$$

The next step is technically crucial. Instead of pursuing the difficult task of applying the blocking directly to H, we switch our attention to the partition function and develop a perturbation solution to the problem.

Following (8.81), we write

$$Z(\boldsymbol{K}_L, N) = \sum_{\{S_I\}} \exp\left[-\beta H(\boldsymbol{K}_L, \{S_I\})\right] \tag{2a}$$

$$= \sum_{\{S_I\}} \sum_{\{\alpha_I\}} \exp\left[-\beta H(\boldsymbol{K}, \{S_I, \alpha_I\})\right] \tag{2}$$

where

$$\sum_{\{S_I\}} \sum_{\{\alpha_I\}} \equiv \sum_{S_1=-1}^{1} \cdots \sum_{S_N=-1}^{1} \sum_{\alpha_1=1}^{4} \cdots \sum_{\alpha_N=1}^{4} \tag{2b}$$

Comparing (2a) and (2) gives

$$\exp\left[-\beta H(\boldsymbol{K}_L, \{S_I\})\right] = \sum_{\{\alpha_I\}} \exp\left[-\beta H(\boldsymbol{K}, \{S_I, \alpha_I\})\right] \tag{2c}$$

Now, the blocking is equivalent to absorbing the internal degrees of freedom into the coupling constants. This can be done by the averaging operation using the "internal" partition function

$$Z_0(\boldsymbol{K}, M) = \sum_{\{\alpha_I\}} \exp[-\beta H_0(\boldsymbol{K}, \{S_I, \alpha_I\})] \tag{8a}$$

$$= \sum_{\{\alpha_I\}} \exp\left(K \sum_{I=1}^{N} \sum_{i,j \in I} s_i s_j\right)$$

$$= \sum_{\{\alpha_I\}} \exp\left[K \sum_{I=1}^{N} \left(s_1^I s_2^I + s_2^I s_3^I + s_3^I s_1^I\right)\right]$$

$$= \prod_{I=1}^{M} \sum_{\alpha_I=1}^{4} \exp\left[K\left(s_1^I s_2^I + s_2^I s_3^I + s_3^I s_1^I\right)\right]$$

where $M = \dfrac{N}{3}$ is the number of blocks.

Since all I evaluates to the same value, we have

$$Z_0(\boldsymbol{K}, M) = \left(\sum_{\alpha=1}^{4} \exp\left(K \mathcal{I}_\alpha\right)\right)^M = [Z_0(K)]^M \tag{8b}$$

where

$$\mathcal{I}_\alpha = (s_1 s_2 + s_2 s_3 + s_3 s_1)_\alpha \tag{8c}$$

and

$$Z_0(K) = \sum_{\alpha=1}^{4} \exp\left(K \mathcal{I}_\alpha\right) \tag{8d}$$

is the internal partition function for a single 3-spin block.

Using

$s_1\, s_2\, s_3$	↓ ↑ ↑	↑ ↓ ↑	↑ ↑ ↓	↑ ↑ ↑	↑ ↓ ↓	↓ ↑ ↓	↓ ↓ ↑	↓ ↓ ↓
(S_I, α_I)	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(−1, 1)	(−1, 2)	(−1, 3)	(−1, 4)
$\mathcal{I}_\alpha$	−1	−1	−1	3	−1	−1	−1	3

(8e)

we have

$$Z_0(K) = 3\, e^{-K} + e^{3K} \tag{9}$$

which is the same for both $S_I = \pm 1$.

Average over the internal degrees of freedom are defined as

$$\langle A(\{S_I\})\rangle = \frac{1}{Z_0(K, M)} \sum_{\{\alpha_I\}} A(\{S_I, \alpha_I\}) \exp[-\beta H_0(\boldsymbol{K}, \{S_I, \alpha_I\})] \tag{6}$$

$$= \frac{1}{Z_0} \sum_{\{\alpha_I\}} A\, e^{-\beta H_0} \qquad \text{[Abbreviated form for clarity.]}$$

From (2c) & (3), we have

$$\exp[-\beta H(\boldsymbol{K}_L, \{S_I\})] = \sum_{\{\alpha_I\}} \exp\left[-\beta H(\boldsymbol{K}, \{S_I, \alpha_I\})\right]$$

$$= \sum_{\{\alpha_I\}} \exp\left[-\beta H_0(\boldsymbol{K}, \{S_I, \alpha_I\})\right] \exp\left[-\beta V(\boldsymbol{K}, \{S_I, \alpha_I\})\right]$$

$$= Z_0(K, M) \left\langle \exp\left[-\beta V(\boldsymbol{K}, \{S_I\})\right] \right\rangle$$

$$\approx Z_0(K, M)\, \exp\left\{ -\beta \left[\langle V\rangle - \frac{1}{2}\beta \left(\langle V^2\rangle - \langle V\rangle^2 \right) + \ldots \right] \right\} \tag{7}$$

where we have used the cumulant expansion [see (4.18) with $i\,k \to -\beta$] to obtain the last expression.

Taking the logarithm of (7) gives

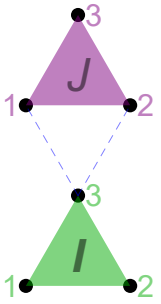
$$H(\boldsymbol{K}_L, \{S_I\}) \approx -\frac{1}{\beta} M \ln Z_0(K) + \langle V\rangle - \frac{1}{2}\beta \left(\langle V^2\rangle - \langle V\rangle^2 \right) + \ldots \tag{11a}$$

which is the promised perturbation formula for the block Hamiltonian.

We now turn to the task of evaluating $\langle V\rangle$.

By symmetry, the interaction between nearest neighbor blocks are all the same and given, for example, by

$$v_{I\,J} = S_I\, S_J\, s_3^I \left(s_1^J + s_2^J \right) \tag{7a}$$



Together with (1b), (5) can now be written as

$$\beta V(\boldsymbol{K}, \{S_I, \sigma_I\}) = -K \sum_{(I, J)} v_{I\,J} - B \sum_I S_I\, \sigma_I$$

so that

$$\beta \langle V\rangle = -K \sum_{(I, J)} S_I\, S_J \left\langle s_3^I \left(s_1^J + s_2^J \right) \right\rangle - B \sum_I S_I \left\langle \left| s_1^I + s_2^I + s_3^I \right| \right\rangle \tag{7b}$$

According to (6), we have

$$\left\langle s_3^I \left(s_1^J + s_2^J\right)\right\rangle = \frac{1}{[\,Z_0(K)\,]^2} \sum_{\{\alpha_I, \alpha_J\}} \mathcal{I}_{\alpha_I \alpha_J} \exp\left[K\left(\mathcal{I}_{\alpha_I} + \mathcal{I}_{\alpha_J}\right)\right] \tag{7c}$$

$$\left\langle \,\middle|\, s_1^I + s_2^I + s_3^I \,\middle|\,\right\rangle = \frac{1}{Z_0(K)} \sum_{\{\alpha_I\}} \sigma_{\alpha_I} \exp\left(K\, \mathcal{I}_{\alpha_I}\right) \tag{7d}$$

where

$$\mathcal{I}_{\alpha_I \alpha_J} = \left[\, s_3^I \left(s_1^J + s_2^J\right)\right]_{\alpha_I,\, \alpha_J} \qquad \sigma_{\alpha_I} = \left|\, s_1^I + s_2^I + s_3^I \,\right|_{\alpha_I} \tag{7e}$$

(7c-d) can be evaluated using the following table obtained using Tables (1c) & (8e).

α_I	α_J	$\mathcal{I}_{\alpha_I \alpha_J}$	$\mathcal{I}_{\alpha_I}$	$\mathcal{I}_{\alpha_J}$	σ_{α_I}
1	1	$\uparrow(\downarrow+\uparrow)=0$	−1	−1	1
1	2	$\uparrow(\uparrow+\downarrow)=0$	−1	−1	
1	3	$\uparrow(\uparrow+\uparrow)=2$	−1	−1	
1	4	$\uparrow(\uparrow+\uparrow)=2$	−1	3	
2	1	$\uparrow(\downarrow+\uparrow)=0$	−1	−1	1
2	2	$\uparrow(\uparrow+\downarrow)=0$	−1	−1	
2	3	$\uparrow(\uparrow+\uparrow)=2$	−1	−1	
2	4	$\uparrow(\uparrow+\uparrow)=2$	−1	3	
3	1	$\downarrow(\downarrow+\uparrow)=0$	−1	−1	1
3	2	$\downarrow(\uparrow+\downarrow)=0$	−1	−1	
3	3	$\downarrow(\uparrow+\uparrow)=-2$	−1	−1	
3	4	$\downarrow(\uparrow+\uparrow)=-2$	−1	3	
4	1	$\uparrow(\downarrow+\uparrow)=0$	3	−1	3
4	2	$\uparrow(\uparrow+\downarrow)=0$	3	−1	
4	3	$\uparrow(\uparrow+\uparrow)=2$	3	−1	
4	4	$\uparrow(\uparrow+\uparrow)=2$	3	3	

(7f)

Thus,

$$\left\langle s_3^I \left(s_1^J + s_2^J\right)\right\rangle = \frac{2\left(e^{-2K} + e^{2K}\right) + 2\left(e^{2K} + e^{6K}\right)}{\left(3\,e^{-K} + e^{3K}\right)^2} = 2\left(\frac{e^{-K} + e^{3K}}{3\,e^{-K} + e^{3K}}\right)^2$$

$$\left\langle \,\middle|\, s_1^I + s_2^I + s_3^I \,\middle|\,\right\rangle = \frac{3\left(e^{-K} + e^{3K}\right)}{3\,e^{-K} + e^{3K}}$$

so that (7b) becomes

$$\langle V\rangle = -2K\left(\frac{e^{-K} + e^{3K}}{3\,e^{-K} + e^{3K}}\right)^2 \sum_{I \neq J} S_I\, S_J - B\,\frac{3\left(e^{-K} + e^{3K}\right)}{3\,e^{-K} + e^{3K}} \sum_I S_I \tag{7g}$$

Keeping terms up to $O(K)$, (11a) becomes

$$H(\boldsymbol{K}_L, \{S_I\}) = -\frac{1}{\beta} M \ln Z_0(K) - 2K\left(\frac{e^{-K} + e^{3K}}{3\,e^{-K} + e^{3K}}\right)^2 \sum_{I \neq J} S_I\, S_J \tag{11}$$

$$-B\,\frac{3\left(e^{-K} + e^{3K}\right)}{3\,e^{-K} + e^{3K}} \sum_I S_I + O\left(K^2\right)$$

Comparing with the renormalized block form

$$H(\boldsymbol{K}_L, \{S_I\}) = -K_L \sum_{(I,J)} S_I\, S_J - B_L \sum_I S_I \qquad S_I = \pm 1 \tag{12}$$

we have

$$K_L = 2K\left(\frac{e^{-K}+e^{3K}}{3e^{-K}+e^{3K}}\right)^2 \tag{13}$$

$$B_L = B\,\frac{3\left(e^{-K}+e^{3K}\right)}{3e^{-K}+e^{3K}} \tag{14}$$

The transform matrix is [see §Code]

$$\mathbb{A} = \left(\begin{array}{cc} \frac{\partial K_L}{\partial K} & \frac{\partial K_L}{\partial B} \\ \frac{\partial B_L}{\partial K} & \frac{\partial B_L}{\partial B} \end{array}\right)_{\boldsymbol{K}=\boldsymbol{K}^*} = \left(\begin{array}{cc} \frac{2\left(1+e^{4K}\right)\left(3+e^{8K}+4e^{4K}(1+4K)\right)}{\left(3+e^{4K}\right)^3} & 0 \\ \frac{24Be^{4K}}{\left(3+e^{4K}\right)^2} & \frac{3\left(1+e^{4K}\right)}{3+e^{4K}} \end{array}\right)_{\boldsymbol{K}=\boldsymbol{K}^*} \tag{14a}$$

Fixed points are solutions of [see (13-4)]

$$K^* = 2K\left(\frac{e^{-K^*}+e^{3K^*}}{3e^{-K^*}+e^{3K^*}}\right)^2 \tag{15a}$$

$$B^* = B^*\,\frac{3\left(e^{-K^*}+e^{3K^*}\right)}{3e^{-K^*}+e^{3K^*}} \tag{15b}$$

(15b) gives

$$B^* = 0$$

while (15a) gives

$$K^* = 0$$

or

$$\frac{1}{2} = \left(\frac{e^{-K^*}+e^{3K^*}}{3e^{-K^*}+e^{3K^*}}\right)^2 \tag{15}$$

There are only one real and positive solution to (15), namely,

$$K_f\beta = \frac{1}{4}\ln\left(1+2\sqrt{2}\right) \approx 0.3356 \tag{15a}$$

Thus, there are two fixed points

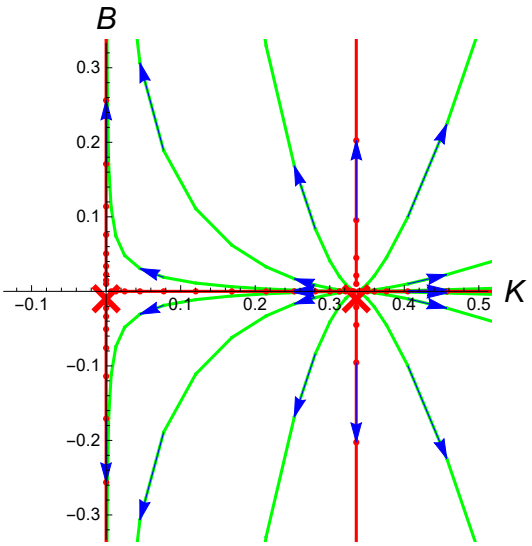
$$\boldsymbol{K}_1^* = \left(\begin{array}{c} K_1^* \\ B_1^* \end{array}\right) = \left(\begin{array}{c} 0 \\ 0 \end{array}\right) \qquad \text{and} \qquad \boldsymbol{K}_2^* = \left(\begin{array}{c} K_2^* \\ B_2^* \end{array}\right) \approx \left(\begin{array}{c} 0.3356 \\ 0 \end{array}\right) \tag{15b}$$

with

$$\mathbb{A}\,|_{\boldsymbol{K}=\boldsymbol{K}_1^*} = \left(\begin{array}{cc} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{array}\right) \qquad \mathbb{A}\,|_{\boldsymbol{K}=\boldsymbol{K}_2^*} \approx \left(\begin{array}{cc} 1.62352 & 0 \\ 0 & 2.12132 \end{array}\right) \tag{15c}$$

which means the eigenvectors are along the K & B directions in both cases.

Flows in the (K, B) space are shown in the graph below. Transformations (13-4) are shown as dots, which are connected by lines for better visualization. Fixed points are shown by red crosses and eigen-curves by red lines.



Therefore, for $\boldsymbol{K}_1^*$,

$$\lambda_K = 0.5 \qquad \lambda_B = 1.5$$
$$\delta K_L = 0.5\, \delta K \qquad \delta K_L = 1.5\, \delta K \tag{16}$$

For $\boldsymbol{K}_2^*$,

$$\lambda_K = 1.62352 \qquad \lambda_B = 2.12132$$
$$\delta K_L = 1.62352\, \delta K \qquad \delta K_L = 2.12132\, \delta K \tag{17}$$

Note that (16-7) can also be obtained directly from expanding (13-4) about the fixed points, as was done by Reichl.

Note that if we keep $J = \text{const}$, then $K = \beta J = \dfrac{J}{k_B T}$ is a temperature parameter.

The fixed point $\tilde{\boldsymbol{K}}_1^*$ thus represent $\beta = 0$ or $T \to \infty$.

Since

$$\delta K = -\frac{J}{k_B T^2}\, \delta T \xrightarrow[T \to T_C]{} \frac{J}{k_B T_C}(-\epsilon) \qquad \epsilon = \frac{T - T_C}{T_C}$$

the Widom scaling indices [cf. (8.96-7) & (8.38)] are, for $\boldsymbol{K}_2^*$,

$$p = \frac{\lambda_K}{2 \ln \sqrt{3}} \approx 0.441101 \qquad q = \frac{\lambda_B}{2 \ln \sqrt{3}} \approx 0.684535 \tag{18}$$

The critical exponents are [see (8.39,42,46 & 50)]

$$\alpha = 2 - \frac{1}{p} \approx -0.267 \qquad \beta = \frac{1-q}{p} \approx 0.715$$
$$\delta = \frac{2q-1}{p} \approx 0.837 \qquad \delta = \frac{q}{1-q} \approx 2.169 \tag{19}$$

(17) should be compared with the exact solution is

$$(\lambda_K)_{\text{exact}} = 1.73 \qquad (\lambda_B)_{\text{exact}} = 2.80$$

Improvement can be achieved by including higher order terms in (11).

Finally, there is a more serious problem: since the $\boldsymbol{K}_2^*$ fixed point is repulsive in all directions, it cannot be reached by increasing ξ and hence not qualified to be a critical point. The results (18-9) are therefore, straightly speaking, worthless. This could be fixed only by increasing the dimensions of the parameter space $\boldsymbol{K}$ through the introduction of new types of interactions into the Hamiltonian [

see, e.g., the S^4 mode]. Having done so, the result in this section can be taken as its projection onto the (B, K) subspace.

Code

```
A := Outer[∂_{#2} #1 &, {KL, BL}, {K, B}]
```

$$\text{KL} := 2\,K \left(\frac{e^{-K}+e^{3K}}{3\,e^{-K}+e^{3K}}\right)^2$$

$$\text{BL} := 3\,B\,\frac{e^{-K}+e^{3K}}{3\,e^{-K}+e^{3K}}$$

```
A // Simplify // MatrixForm
```

$$\begin{pmatrix} \frac{2\left(1+e^{4K}\right)\left(3+e^{8K}+4\,e^{4K}\,(1+4\,K)\right)}{\left(3+e^{4K}\right)^3} & 0 \\ \frac{24\,B\,e^{4K}}{\left(3+e^{4K}\right)^2} & \frac{3\left(1+e^{4K}\right)}{3+e^{4K}} \end{pmatrix}$$

```
(* K_f: general solutions *)
solf = Assuming[β > 0, Solve[((e^-K + e^3K)/(3 e^-K + e^3K))^2 == 1/2, K]]
```

$$\left\{\left\{K \to \text{ConditionalExpression}\left[2\,i\,\pi\,C[1] + \text{Log}\left[-\left(1-2\sqrt{2}\right)^{1/4}\right], C[1] \in \text{Integers}\right]\right\},\right.$$
$$\left\{K \to \text{ConditionalExpression}\left[2\,i\,\pi\,C[1] + \text{Log}\left[-i\left(1-2\sqrt{2}\right)^{1/4}\right], C[1] \in \text{Integers}\right]\right\},$$
$$\left\{K \to \text{ConditionalExpression}\left[2\,i\,\pi\,C[1] + \text{Log}\left[i\left(1-2\sqrt{2}\right)^{1/4}\right], C[1] \in \text{Integers}\right]\right\},$$
$$\left\{K \to \text{ConditionalExpression}\left[2\,i\,\pi\,C[1] + \frac{1}{4}\left(i\,\pi + \text{Log}\left[-1+2\sqrt{2}\right]\right), C[1] \in \text{Integers}\right]\right\},$$
$$\left\{K \to \text{ConditionalExpression}\left[2\,i\,\pi\,C[1] + \text{Log}\left[-i\left(1+2\sqrt{2}\right)^{1/4}\right], C[1] \in \text{Integers}\right]\right\},$$
$$\left\{K \to \text{ConditionalExpression}\left[2\,i\,\pi\,C[1] + \text{Log}\left[i\left(1+2\sqrt{2}\right)^{1/4}\right], C[1] \in \text{Integers}\right]\right\},$$
$$\left\{K \to \text{ConditionalExpression}\left[2\,i\,\pi\,C[1] + \frac{1}{4}\text{Log}\left[1+2\sqrt{2}\right], C[1] \in \text{Integers}\right]\right\},$$
$$\left.\left\{K \to \text{ConditionalExpression}\left[i\,\pi + 2\,i\,\pi\,C[1] + \frac{1}{4}\text{Log}\left[1+2\sqrt{2}\right], C[1] \in \text{Integers}\right]\right\}\right\}$$

```
(* K_f with -π ≤ Θ < π *)
Kf = K /. solf /. C[1] → 0
```

$$\left\{\text{Log}\left[-\left(1-2\sqrt{2}\right)^{1/4}\right], \text{Log}\left[-i\left(1-2\sqrt{2}\right)^{1/4}\right],\right.$$
$$\text{Log}\left[i\left(1-2\sqrt{2}\right)^{1/4}\right], \frac{1}{4}\left(i\,\pi + \text{Log}\left[-1+2\sqrt{2}\right]\right), \text{Log}\left[-i\left(1+2\sqrt{2}\right)^{1/4}\right],$$
$$\left.\text{Log}\left[i\left(1+2\sqrt{2}\right)^{1/4}\right], \frac{1}{4}\text{Log}\left[1+2\sqrt{2}\right], i\,\pi + \frac{1}{4}\text{Log}\left[1+2\sqrt{2}\right]\right\}$$

```
(* K_f:  numerical *)
Kf // N
```

$$\{0.150864 - 2.35619\,i,\ 0.150864 - 0.785398\,i,\ 0.150864 + 2.35619\,i,\ 0.150864 + 0.785398\,i,$$
$$0.335614 - 1.5708\,i,\ 0.335614 + 1.5708\,i,\ 0.335614,\ 0.335614 + 3.14159\,i\}$$

```
(* K_f: real solutions only *)
solR = NSolve[(e^(3 K) + e^-K)/(e^(3 K) + 3 e^-K))^2 == 1/2, K, Reals]
```

```
{{K → 0.335614}}
```

```
((A /. B → 0) /. solf[[7]] /. C[1] → 0) // Simplify;
```

```
(* 𝔸 *)
A // MatrixForm
```

$$\begin{pmatrix} \frac{2\left(e^{-K}+e^{3K}\right)^2}{\left(3e^{-K}+e^{3K}\right)^2} - \frac{4\left(e^{-K}+e^{3K}\right)^2\left(-3e^{-K}+3e^{3K}\right)K}{\left(3e^{-K}+e^{3K}\right)^3} + \frac{4\left(e^{-K}+e^{3K}\right)\left(-e^{-K}+3e^{3K}\right)K}{\left(3e^{-K}+e^{3K}\right)^2} & 0 \\ -\frac{3B\left(e^{-K}+e^{3K}\right)\left(-3e^{-K}+3e^{3K}\right)}{\left(3e^{-K}+e^{3K}\right)^2} + \frac{3B\left(-e^{-K}+3e^{3K}\right)}{3e^{-K}+e^{3K}} & \frac{3\left(e^{-K}+e^{3K}\right)}{3e^{-K}+e^{3K}} \end{pmatrix}$$

```
(* 𝔸 at K*=0,B*=0 *)
A1 = A /. {B → 0, K → 0};
A1 // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix}$$

```
(* 𝔸 at K*=K_f,B*=0 *)
A2 = (A /. B → 0) /. (solf[[7]] /. C[1] → 0) // Simplify;
A2 // MatrixForm
```

$$\begin{pmatrix} \frac{2\left(10+7\sqrt{2}+\left(5+3\sqrt{2}\right)\text{Log}\left[1+2\sqrt{2}\right]\right)}{\left(2+\sqrt{2}\right)^3} & 0 \\ 0 & \frac{3}{\sqrt{2}} \end{pmatrix}$$

```
A2 // N // MatrixForm
```

$$\begin{pmatrix} 1.62352 & 0. \\ 0. & 2.12132 \end{pmatrix}$$

```
(* scaling indices *)
{p, q} = Log[#]/(2 Log[Sqrt[3]]) & /@ Diagonal[A2]
{p, q} // N
```

$$\left\{\frac{\text{Log}\left[\frac{2\left(10+7\sqrt{2}+\left(5+3\sqrt{2}\right)\text{Log}\left[1+2\sqrt{2}\right]\right)}{\left(2+\sqrt{2}\right)^3}\right]}{\text{Log}[3]}, \frac{\text{Log}\left[\frac{3}{\sqrt{2}}\right]}{\text{Log}[3]}\right\}$$

```
{0.441101, 0.684535}
```

```
(* critical exponents *)
{α, β, γ, δ} = {2 - 1/p, (1 - q)/p, (2 q - 1)/p, q/(1 - q)} // N
```

```
{-0.267053, 0.715176, 0.836702, 2.16993}
```