9.B. Thermodynamics and the Radial Distribution Function

Read §9.A of Reichl's text.

A classical fluid of N particles and volume V is described by the Hamiltonian

$$H^N(\boldsymbol{q}^N, \boldsymbol{p}^N) = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \sum_{(i,j)}^{N(N-1)/2} \mathcal{V}(\boldsymbol{q}_{ij}) \tag{9.1}$$

where the 2nd sum is over the $\frac{1}{2} N(N-1)$ pairs of particles and

$$\boldsymbol{q}^N = (\boldsymbol{q}_1, \; ..., \boldsymbol{q}_N) \qquad\qquad \boldsymbol{p}^N = (\boldsymbol{p}_1, \; ..., \boldsymbol{p}_N)$$

$\boldsymbol{q}_{ij} = \boldsymbol{q}_j - \boldsymbol{q}_i$ = vector pointing from particle i to j.

In general, a 1-body phase function takes the form

$$O_1^N(\boldsymbol{q}^N) = \sum_{i=1}^{N} O(\boldsymbol{q}_i) \tag{9.1a}$$

and a 2-body one

$$O_2^N(\boldsymbol{q}^N) = \sum_{(i,j)}^{N(N-1)/2} O(\boldsymbol{q}_i, \boldsymbol{q}_j) \tag{9.1b}$$

For example, we have

$$K_1^N(\boldsymbol{p}^N) = \sum_{i=1}^{N} \frac{p_i^2}{2m} \tag{9.1c}$$

$$\mathcal{V}_2^N(\boldsymbol{q}^N) = \sum_{(i,j)}^{N(N-1)/2} \mathcal{V}(\boldsymbol{q}_{ij}) \tag{9.1d}$$

The thermal averages of these functions are given by

$$\langle O_1^N \rangle = \frac{1}{Z_N} \int d\,\boldsymbol{q}^N \int d\,\boldsymbol{p}^N \, e^{-\beta H^N} \, O_1^N(\boldsymbol{q}^N) \tag{9.2a}$$

$$\langle O_2^N \rangle = \frac{1}{Z_N} \int d\,\boldsymbol{q}^N \int d\,\boldsymbol{p}^N \, e^{-\beta H^N} \, O_2^N(\boldsymbol{q}^N) \tag{9.3a}$$

where

$$Z_N = \int d\,\boldsymbol{q}^N \int d\,\boldsymbol{p}^N \, e^{-\beta H^N} \tag{9.2b}$$

is the N-particle partition function.

It is understood that every $\boldsymbol{q}_i$-integral is over V and every $\boldsymbol{p}_i$-integral is over all space.

Using (9.1) on (9.2b) gives

$$Z_N = \left(\prod_{j=1}^{N} \int d\,\boldsymbol{p}_i \, e^{-\beta p_i^2/2m} \right) \int d\,\boldsymbol{q}^N \; e^{-\beta \mathcal{V}_2^N}$$

$$= z^N \, Q_N(V, T) \tag{9.2c}$$

where V is the volume of the system and

$$z = \int d\,\boldsymbol{p} \, e^{-\beta p^2/2m} \tag{9.2d}$$

is the partition function for one free particle and

$$Q_N(V, T) = \int d\,\boldsymbol{q}^N \, e^{-\beta \mathcal{V}_2^N} \qquad\qquad \beta = \frac{1}{k_B T} \tag{9.5}$$

is the configuration integral.

Similarly,

$$\int d\,\boldsymbol{q}^N \int d\,\boldsymbol{p}^N e^{-\beta H^N} O_1^N(\boldsymbol{q}^N)$$
$$= z^N \sum_{i=1}^{N} \int d\,\boldsymbol{q}^N e^{-\beta \mathcal{V}_2^N} O(\boldsymbol{q}_i)$$
$$= z^N N \int d\,\boldsymbol{q}^N e^{-\beta \mathcal{V}_2^N} O(\boldsymbol{q}_1) \qquad \text{[Integral gives same numerical value } \forall\ i. \text{]}$$
$$= z^N C_1^N \int d\,\boldsymbol{q}_1\, O_1(\boldsymbol{q}_1) \int d\,\boldsymbol{q}_2 \ldots \int d\,\boldsymbol{q}_N\, e^{-\beta \mathcal{V}_2^N} \qquad (9.2e)$$

where

$$C_k^n = \frac{n!}{k!\,(n-k)!} = k\text{-combinations of } n \text{ objects.}$$

(9.2a) thus becomes

$$\langle O_1^N \rangle = \frac{C_1^N}{Q_N(V,\,T)} \int d\,\boldsymbol{q}_1\, O(\boldsymbol{q}_1) \int d\,\boldsymbol{q}_2 \ldots \int d\,\boldsymbol{q}_N\, e^{-\beta \mathcal{V}_2^N}$$
$$= \int d\,\boldsymbol{q}_1\, O_1(\boldsymbol{q}_1)\, n_1^N(\boldsymbol{q}_1;\, V,\, T) \qquad (9.2)$$

where

$$n_1^N(\boldsymbol{q}_1;\, V,\, T) = \frac{C_1^N}{Q_N(V,\,T)} \int d\,\boldsymbol{q}_2 \ldots \int d\,\boldsymbol{q}_N\, e^{-\beta \mathcal{V}_2^N} \qquad (9.4a)$$

is the 1-body reduced distribution function. As can be seen from (9.2),

$n_1^N(\boldsymbol{q}_1;\, V,\, T)\,\Delta\,\boldsymbol{q}_1$ = probability of finding a particle inside $\Delta\boldsymbol{q}_1$ at $\boldsymbol{q}_1$.

Note that contributions from the kinetic energy are cancelled out because we can write

$$e^{-\beta H^N} = e^{-\beta K_1^N}\, e^{-\beta \mathcal{V}_2^N}$$

Since this is not valid for a quantum system, what follows applies only to classical fluids.

The 2-body version of (9.2e) is easily found to be

$$\int d\,\boldsymbol{q}^N \int d\,\boldsymbol{p}^N e^{-\beta H^N} O_2^N(\boldsymbol{q}^N)$$
$$= z^N C_2^N \int d\,\boldsymbol{q}_1 \int d\,\boldsymbol{q}_2\, O(\boldsymbol{q}_{12}) \int d\,\boldsymbol{q}_3 \ldots \int d\,\boldsymbol{q}_N\, e^{-\beta \mathcal{V}_2^N} \qquad (9.3b)$$

(9.3a) thus becomes

$$\langle O_2^N \rangle = \frac{C_2^N}{Q_N(V,\,T)} \int d\,\boldsymbol{q}_1 \int d\,\boldsymbol{q}_2\, O(\boldsymbol{q}_{12}) \int d\,\boldsymbol{q}_3 \ldots \int d\,\boldsymbol{q}_N\, e^{-\beta \mathcal{V}_2^N}$$
$$= \frac{1}{2} \int d\,\boldsymbol{q}_1 \int d\,\boldsymbol{q}_2\, O\,(\boldsymbol{q}_{12})\, n_2^N\,(\boldsymbol{q}_1,\, \boldsymbol{q}_2\,;\, V,\, T) \qquad (9.3)$$

where

$$n_2^N(\boldsymbol{q}_1,\, \boldsymbol{q}_2\,;\, V,\, T) = \frac{2\, C_2^N}{Q_N(V,\,T)} \int d\,\boldsymbol{q}_3 \ldots \int d\,\boldsymbol{q}_N\, e^{-\beta \mathcal{V}_2^N} \qquad (9.4b)$$

is the 2-body reduced distribution function. As can be seen from (9.3),

$n_2^N(\boldsymbol{q}_1,\, \boldsymbol{q}_2\,;\, V,\, T)\,\Delta\,\boldsymbol{q}_1\,\Delta\,\boldsymbol{q}_2$

= probability of finding one particle inside $\Delta\boldsymbol{q}_1$ at $\boldsymbol{q}_1$

& another inside $\Delta\boldsymbol{q}_2$ at $\boldsymbol{q}_2$, simultaneously.

The prefactor $\frac{1}{2}$ in (9.3) is to cancel the double counting.

In general,

$$\langle O_k^N \rangle = \frac{1}{k!} \int d\,\boldsymbol{q}^k\, O\left(\boldsymbol{q}^k\right) n_k^N\left(\boldsymbol{q}^k ; V, T\right) \qquad (9.3c)$$

where

$$n_k^N(\boldsymbol{q}^k ; V, T) = \frac{k!\, C_k^N}{Q_N(V, T)} \int d\,\boldsymbol{q}_{k+1} \ldots \int d\,\boldsymbol{q}_N\, e^{-\beta \mathcal{V}_2^N} \qquad (9.4)$$

is the k-body reduced distribution function. As before,

$$n_k^N(\boldsymbol{q}^k ; V, T)\, \Delta\,\boldsymbol{q}_1 \ldots \Delta\,\boldsymbol{q}_k \qquad (9.4a)$$

= probability of finding one particle inside $\Delta\boldsymbol{q}_1$ at $\boldsymbol{q}_1$,
one inside $\Delta\boldsymbol{q}_2$ at $\boldsymbol{q}_2$, ... , and one inside $\Delta\boldsymbol{q}_k$ at $\boldsymbol{q}_k$, simultaneously.

The prefactor $\frac{1}{k!}$ in (9.3c) is necessary since permutations among $\boldsymbol{q}_1$, ..., $\boldsymbol{q}_k$ correspond to relabellings of the same set of particles, and hence describing the same physical configuration.

With the help of (9.5), we integrate (9.4) to get

$$\int d\,\boldsymbol{q}^k\, n_k^N(\boldsymbol{q}^k ; V, T) = \frac{k!\, C_k^N}{Q_N(V, T)} \int d\,\boldsymbol{q}_1 \ldots \int d\,\boldsymbol{q}_N\, e^{-\beta \mathcal{V}_2^N}$$

$$= k!\, C_k^N = \frac{N!}{(N-k)!} \qquad (9.6)$$

As an example, consider the internal energy, which is defined as the thermal average of the Hamiltonian [see (7.43-4) of §7.D.1] :

$$U(V, T, N) = \langle H^N \rangle = \langle K_1^N \rangle + \langle \mathcal{V}_2^N \rangle \qquad (9.7a)$$

Using (9.2c), we have

$$\langle K_1^N \rangle = \frac{1}{z^N} \int d\,\boldsymbol{p}^N\, K_1^N\, e^{-\beta K_1^N}$$

$$= \frac{1}{z^N} \sum_{j=1}^{N} \left(\int d\,\boldsymbol{p}_j \frac{p_j^2}{2\,m} e^{-\beta p_j^2/2\,m} \prod_{i \neq j} \int d\,\boldsymbol{p}_i\, e^{-\beta p_i^2/2\,m} \right)$$

$$= \frac{N}{z} \int d\,\boldsymbol{p} \frac{p^2}{2\,m} e^{-\beta p^2/2\,m} \qquad \text{[(9.2d) used.]}$$

$$= \frac{N}{z} 4\,\pi \int_0^\infty d\,p \frac{p^4}{2\,m} e^{-\beta p^2/2\,m}$$

$$= N \frac{3}{2\,\beta} \qquad \text{[See §Code]}$$

$$= \frac{3}{2} N\, k_B\, T \qquad (9.7b)$$

where

$$z = 4\,\pi \int_0^\infty d\,p\, p^2\, e^{-\beta p^2/2\,m} = \left(\frac{2\,\pi\,m}{\beta}\right)^{3/2} \qquad (9.7c)$$

With (9.3), we can write (9.7a) as

$$U(V, T, N) = \frac{3}{2} N\, k_B\, T + \frac{1}{2} \int d\,\boldsymbol{q}_1 \int d\,\boldsymbol{q}_2\, \mathcal{V}(\boldsymbol{q}_{12})\, n_2^N(\boldsymbol{q}_1, \boldsymbol{q}_2 ; V, T) \qquad (9.7)$$

Consider now the case where $\mathcal{V}(\boldsymbol{q}_{ij})$ is a central (or spherically symmetric) potential, i.e.,

$$\mathcal{V}(\boldsymbol{q}_{ij}) = \mathcal{V}(q_{ij}) \qquad q_{ij} = |\,\boldsymbol{q}_{ij}\,| \qquad (9.8a)$$

Introducing the center-of-mass coordinates

$$\boldsymbol{Q}_{ij} = \frac{1}{2} (\boldsymbol{q}_j + \boldsymbol{q}_i) \qquad (9.8b)$$

we have

$$\boldsymbol{q}_i = \boldsymbol{Q}_{ij} - \frac{1}{2}\boldsymbol{q}_{ij} \qquad\qquad \boldsymbol{q}_j = \boldsymbol{Q}_{ij} + \frac{1}{2}\boldsymbol{q}_{ij} \qquad\qquad (9.8c)$$

The Jacobian of the transformation is

$$J = \frac{\partial(\boldsymbol{q}_i, \boldsymbol{q}_j)}{\partial(\boldsymbol{Q}_{ij}, \boldsymbol{q}_{ij})} = \begin{vmatrix} \frac{\partial \boldsymbol{q}_i}{\partial \boldsymbol{Q}_{ij}} & \frac{\partial \boldsymbol{q}_i}{\partial \boldsymbol{q}_{ij}} \\ \frac{\partial \boldsymbol{q}_j}{\partial \boldsymbol{Q}_{ij}} & \frac{\partial \boldsymbol{q}_j}{\partial \boldsymbol{q}_{ij}} \end{vmatrix} = \begin{vmatrix} \boldsymbol{I} & -\frac{1}{2}\boldsymbol{I} \\ \boldsymbol{I} & \frac{1}{2}\boldsymbol{I} \end{vmatrix} = \begin{vmatrix} 2\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{I} & \frac{1}{2}\boldsymbol{I} \end{vmatrix} = 1$$

where Gaussian elimination was used to obtain the 2nd to last expression.

Hence, (9.3) can be written as

$$\begin{aligned}\langle \mathcal{V}_2^N \rangle &= \frac{1}{2}\int d\,\boldsymbol{q}_{12} \int d\,\boldsymbol{Q}_{12}\, \mathcal{V}(\boldsymbol{q}_{12})\, n_2^N(\boldsymbol{q}_{12}\,;V,\,T) \\ &= \frac{1}{2} V \int d\,\boldsymbol{q}_{12}\, \mathcal{V}(\boldsymbol{q}_{12})\, n_2^N(\boldsymbol{q}_{12}\,;V,\,T) \qquad\qquad (9.8d)\end{aligned}$$

For central potentials, (9.4b) becomes

$$n_2^N(\boldsymbol{q}_{12}\,;V,\,T) = \frac{2\,C_2^N}{Q_N(V,\,T)}\, e^{-\beta\mathcal{V}(q_{12})} \int d\,\boldsymbol{q}_3 \ldots \int d\,\boldsymbol{q}_N\, e^{-\beta\mathcal{V}_2^{N-2}}$$

where $\mathcal{V}_2^{N-2}$ is a sum of

1. interactions between particles 3, 4, ..., N.
2. interactions between particle 1 and particles 3, 4, ..., N.
3. interactions between particle 2 and particles 3, 4, ..., N.

For short-range potentials, boundary effects can be neglected. Contributions from items 2 & 3 can then only depend on q_{12} since taken individually, they are equivalent.

We therefore have

$$n_2^N(\boldsymbol{q}_{12}\,;V,\,T) = n_2^N\,(q_{12}\,;V,\,T)$$

and (9.8d) becomes

$$\langle \mathcal{V}_2^N \rangle = \frac{1}{2} V\, 4\pi \int d\,q\; q^2\, \mathcal{V}(q)\, n_2^N(q\,;V,\,T) \qquad\qquad (9.8e)$$

If we have only 1 particle, the probability of finding it inside $\Delta\,\boldsymbol{q}$ any where is $\frac{\Delta\,\boldsymbol{q}}{V}$.

If we have N non-interacting particles, then this probability becomes $N\frac{\Delta\,\boldsymbol{q}}{V} = n\,\Delta\,\boldsymbol{q}$, where $n = \frac{N}{V}$ is the average particle density.

Now,

$$n_2^N(\boldsymbol{q}_1,\,\boldsymbol{q}_2\,;V,\,T)\,\Delta\,\boldsymbol{q}_1\,\Delta\,\boldsymbol{q}_2 = P_1\,P_{1|2} \qquad\qquad (9.8f)$$

where

P_1 = probability of particle 1 in $\Delta\,\boldsymbol{q}_1$ at $\boldsymbol{q}_1$.

$P_{1|2}$ = conditional probability of, given particle 1, finding particle 2 in $\Delta\,\boldsymbol{q}_2$ at $\boldsymbol{q}_2$.

For non-interacting particles,

$$P_1 = \frac{N}{V}\Delta\,\boldsymbol{q}_1$$

$$P_{1|2} = \frac{N-1}{V}\Delta\,\boldsymbol{q}_2$$

$$\rightarrow \qquad n_2^N(\boldsymbol{q}_1,\,\boldsymbol{q}_2\,;V,\,T) = \frac{N(N-1)}{V^2} \approx n^2 \qquad \text{for} \quad N >> 1 \qquad\qquad (9.8g)$$

This suggests writing

$$n_2^N(q\,;V,\,T) \equiv \frac{N(N-1)}{V^2}\, g_2^N(q,\,V,\,T) \qquad (9.8h)$$
$$\approx n^2\, g_2^N(q,\,V,\,T)$$

where $g_2^N(q,\,V,\,T)$ is a correlation function known as the radial distribution function.

(9.7) thus becomes

$$U(V,\,T,\,N) = \frac{3}{2} N\, k_B\, T + \frac{1}{2}\frac{N^2}{V} 4\,\pi \int d\,q\; q^2\, \mathcal{V}(q)\, g_2^N(q,\,V,\,T) \qquad (9.8)$$

Comparing (9.8h) with (9.8g), we have

$g_2^N(q,\,V,\,T) = 1$ for non-interacting particles (with no correlation)

Thus, we expect

$$g_2^N(q,\,V,\,T) \xrightarrow[q>>R]{} 1 \qquad (9.8i)$$

where R is the range of $\mathcal{V}$.

The definition (9.8h) also means that [see (9.8f)]

$$n\, g_2^N(q,\,V,\,T)\, 4\,\pi\, q^2\, \Delta\, q \qquad (9.8j)$$

= probability of finding a particle in the spherical shell of thickness $\Delta\, q$ and radius q centered at a given particle at the origin.

Exercise 9.1

The static density correlation function for the equilibrium fluid discussed above can be written as

$$C_{n\,n}(\boldsymbol{r}) = \frac{1}{N} \int d\,\boldsymbol{r}'\, \langle\, n(\boldsymbol{r}' + \boldsymbol{r})\, n(\boldsymbol{r}')\, \rangle \qquad (1a)$$

where the density phase function is given by

$$n(\boldsymbol{r}) = \sum_{i=1}^{N} \delta(\boldsymbol{r} - \boldsymbol{q}_i) \qquad (1b)$$

The Fourier transform of $C_{n\,n}(\boldsymbol{r})$ is called the structural factor:

$$S_{n\,n}(\boldsymbol{k}) = \int d\,\boldsymbol{r}\, e^{-i\,\boldsymbol{k}\cdot\boldsymbol{r}}\, C_{n\,n}(\boldsymbol{r}) \qquad (1c)$$

Show that in the thermodynamic limit

$$N,\, V \to \infty \qquad \text{with} \qquad \frac{N}{V} = n$$

we can write

$$S_{n\,n}(\boldsymbol{k}) = 1 + (2\,\pi)^3\, n\, \delta(\boldsymbol{k}) + \frac{4\,\pi}{k}\, n \int_0^\infty d\,q\, \left[g_2^N(q) - 1 \right] q^2 \sin k\, q \qquad (1d)$$

Answer

Putting (1a) into (1c) gives

$$S_{n\,n}(\boldsymbol{k}) = \frac{1}{N} \int d\,\boldsymbol{r}\, e^{-i\,\boldsymbol{k}\cdot\boldsymbol{r}} \int d\,\boldsymbol{r}'\, \langle\, n(\boldsymbol{r}' + \boldsymbol{r})\, n(\boldsymbol{r}')\, \rangle$$
$$= \frac{1}{N} \sum_{i,j=1}^{N} \int d\,\boldsymbol{r}\, e^{-i\,\boldsymbol{k}\cdot\boldsymbol{r}} \int d\,\boldsymbol{r}'\, \langle\, \delta(\boldsymbol{r}' + \boldsymbol{r} - \boldsymbol{q}_i)\, \delta(\boldsymbol{r}' - \boldsymbol{q}_j)\, \rangle \qquad \text{[(1b) used.]}$$

$$= \frac{1}{N} \sum_{i,j=1}^{N} \int d\boldsymbol{r}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \langle \delta(\boldsymbol{q}_j + \boldsymbol{r} - \boldsymbol{q}_i) \rangle$$

$$= \frac{1}{N} \sum_{i,j=1}^{N} \left\langle e^{i\boldsymbol{k}\cdot(\boldsymbol{q}_j - \boldsymbol{q}_i)} \right\rangle \qquad (1)$$

$$= 1 + \frac{1}{N} \sum_{\substack{i,j=1 \\ (i\neq j)}}^{N} \left\langle e^{i\boldsymbol{k}\cdot(\boldsymbol{q}_j - \boldsymbol{q}_i)} \right\rangle$$

$$= 1 + \frac{2}{N} \sum_{(i,j)}^{N(N-1)/2} \left\langle e^{i\boldsymbol{k}\cdot(\boldsymbol{q}_j - \boldsymbol{q}_i)} \right\rangle \qquad (2a)$$

Using (9.3), we have

$$S_{nn}(\boldsymbol{k}) = 1 + \frac{1}{N} \int d\boldsymbol{q}_1 \int d\boldsymbol{q}_2\, e^{i\boldsymbol{k}\cdot(\boldsymbol{q}_2 - \boldsymbol{q}_1)}\, n_2^N(\boldsymbol{q}_1, \boldsymbol{q}_2; V, T)$$

$$= 1 + \frac{V}{N} \int d\boldsymbol{q}_{12}\, e^{i\boldsymbol{k}\cdot\boldsymbol{q}_{12}}\, n_2^N(\boldsymbol{q}_{12}; V, T)$$

$$= 1 + \frac{N}{V} \int d\boldsymbol{q}_{12}\, e^{i\boldsymbol{k}\cdot\boldsymbol{q}_{12}}\, g_2^N(q_{12}; V, T) \qquad \text{[(9.8h) used.]} \qquad (2)$$

Since $g_2^N \to 1$ for large q [see (9.8i)], the integral in (2) diverges for $\boldsymbol{k} = 0$ as $V \to \infty$.

Using

$$\frac{1}{V} \int d\boldsymbol{q}\, e^{i\boldsymbol{k}\cdot\boldsymbol{q}} = \delta_{\boldsymbol{k}0} = \frac{(2\pi)^3}{V}\, \delta(\boldsymbol{k})$$

we can extract this singularity outside the integral and write (2) as

$$S_{nn}(\boldsymbol{k}) = 1 + n(2\pi)^3\, \delta(\boldsymbol{k}) + n \int d\boldsymbol{q}_{12}\, e^{i\boldsymbol{k}\cdot\boldsymbol{q}_{12}} \left[g_2^N(q_{12}; V, T) - 1 \right] \qquad (3)$$

$$= 1 + n(2\pi)^3\, \delta(\boldsymbol{k}) + 2\pi n \int_{-1}^{1} d\cos\theta \int dq\, q^2\, e^{ikq\cos\theta} \left[g_2^N(q; V, T) - 1 \right]$$

$$= 1 + n(2\pi)^3\, \delta(\boldsymbol{k}) + 2\pi n \int dq\, q^2\, \frac{e^{ikq} - e^{-ikq}}{ik} \left[g_2^N(q; V, T) - 1 \right]$$

$$= 1 + n(2\pi)^3\, \delta(\boldsymbol{k}) + \frac{4\pi n}{k} \int dq\, q^2\, (\sin kq) \left[g_2^N(q; V, T) - 1 \right] \qquad (4)$$

$S_{nn}(\boldsymbol{k})$ is proportional to the scattering amplitude in the $\boldsymbol{k}$ direction. The $\delta(\boldsymbol{k})$ term thus represents the forward scattering contribution. The result of a neutron scattering experiment is shown in Fig.9.1 in Reichl's text. Also shown is g_2^N obtained from the Fourier inverse of $S_{nn}(\boldsymbol{k})$.

Code

```
z = Assuming[a > 0, ∫_0^∞ p² ⅇ^(-a p²) ⅆp] /. a → β/(2 m);
z = 4 π z // PowerExpand // Simplify
```

$$\frac{2\sqrt{2}\, m^{3/2}\, \pi^{3/2}}{\beta^{3/2}}$$

```
A = Assuming[a > 0, ∫_0^∞ p^4 e^(-a p^2) dp] /. a → β/(2 m);
4 π A/(2 m) // PowerExpand // Simplify
1/z 4 π A/(2 m) // Simplify
```

$$\frac{3\sqrt{2}\ m^{3/2}\ \pi^{3/2}}{\beta^{5/2}}$$

$$\frac{3}{2\beta}$$