

10.B.1. Balance Equations

Read Reich's §10.A.

The general balance equation is derived in Appendix A.

See Landau & Lifshitz, "Fluid Mechanics", Chap.1, for a more intuitive approach to the problem.

10.B.1.0. Introduction

We define **fluid** as a mathematical continuum in which every (fluid) point represents a macroscopically small, but microscopically large spatial region that contains enough number of molecules so that, on the time scale of interest, it can always be taken as in local, temporary, thermal dynamic equilibrium. In other words, every fluid point can be assigned a set of usually time-dependent thermodynamic variables.

From the mechanical point of view, the dynamical state of a fluid is specified by its velocity field $\mathbf{v}(\mathbf{x}, t)$ and mass density $\rho(\mathbf{x}, t)$. Taking the ideal gas as a prototype, its thermodynamical state is specified by any two thermodynamic variables, e.g., $\rho(\mathbf{x}, t)$ & pressure $P(\mathbf{x}, t)$. Note that an **equation of state** (or **constituent equation**) is needed to obtain other thermodynamic quantities.

For a d -D fluid, there are $d + 2$ scalar fields in $\mathbf{v}(\mathbf{x}, t)$, $\rho(\mathbf{x}, t)$ & $P(\mathbf{x}, t)$. Therefore, we need $d + 2$ independent scalar differential equations to describe the dynamics of a fluid. In the following, possible candidates are derived as balance equations due to conservation laws.

10.B.1.1. Mass Balance Equation

Consider a volume $V(t)$ of mass M moving with the fluid. Mass conservation means

$$\frac{dM}{dt} = 0 = \frac{d}{dt} \int_{V(t)} d\mathbf{r} \rho(\mathbf{r}, t) \quad (10.1a)$$

where ρ is the mass density.

The time dependent integration limit can be handled using a coordinate system \mathbf{z} that is "moving" with the fluid [see Appendix A]:

$$\int_{V(t)} d\mathbf{r} \rho(\mathbf{r}, t) = \int_{V_z} d\mathbf{z} J \rho(\mathbf{z}, t)$$

where V_z is $V(t)$ as seen in the moving system, and $J = \det \frac{\partial r_i}{\partial z_j}$ is the Jacobian of the coordinate

transformation.

Using

$$\frac{dJ}{dt} = J \nabla_r \cdot \mathbf{v} \quad [\text{ See (A.12) of Appendix.A. }]$$

we have

$$\begin{aligned} \frac{dM}{dt} &= \frac{d}{dt} \int_{V_z} d\mathbf{z} J \rho \\ &= \int_{V_z} d\mathbf{z} \left(J \frac{d\rho}{dt} + \frac{dJ}{dt} \rho \right) \end{aligned}$$

$$\begin{aligned}
&= \int_{V_z} dz J \left(\frac{d\rho}{dt} + \rho \nabla_r \cdot \mathbf{v} \right) \\
&= \int_{V(t)} dr \left(\frac{d\rho}{dt} + \rho \nabla_r \cdot \mathbf{v} \right) = 0
\end{aligned} \tag{10.1}$$

Since $V(t)$ is arbitrary,

$$\frac{d\rho}{dt} + \rho \nabla_r \cdot \mathbf{v} = 0 \tag{10.2}$$

Using

$$\frac{df[r(t), t]}{dt} = \frac{\partial f}{\partial t} + \frac{dr}{dt} \cdot \nabla_r f = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_r f \tag{10.2a}$$

we have

$$\frac{\partial \rho}{\partial t} + \nabla_r \cdot (\rho \mathbf{v}) = 0 \tag{10.3}$$

where $\mathbf{J} = \rho \mathbf{v}$ is called the **mass current density** or **mass flux**. It also denotes the **momentum density**.

The **mass balance equation** (10.3) is also called the **equation of continuity**. As already stated, it is the direct consequence of mass conservation.

10.B.1.1.1. Digression

Consider the density $\mathcal{F} = \rho f$ of some physical quantity, where f is the quantity per unit mass.

$$\begin{aligned}
\frac{\partial \mathcal{F}}{\partial t} &= \frac{\partial \rho f}{\partial t} = \frac{\partial \rho}{\partial t} f + \rho \frac{\partial f}{\partial t} \\
&= -f \nabla_r \cdot (\rho \mathbf{v}) + \rho \left(\frac{df}{dt} - \mathbf{v} \cdot \nabla_r f \right) && \text{[(10.2a) \& (10.3) used.]} \\
&= \rho \frac{df}{dt} - \nabla_r \cdot (\rho \mathbf{v} f)
\end{aligned}$$

$$\rightarrow \rho \frac{df}{dt} = \frac{\partial \rho f}{\partial t} + \nabla_r \cdot \mathbf{J}_{cf} \tag{10.3a}$$

where $\mathbf{J}_{cf} = \rho \mathbf{v} f = \mathcal{F} \mathbf{v}$ is the **convective flux** of f that is due entirely to the flow of the fluid.

If f is **conserved**, then

$$\frac{df}{dt} = 0 \tag{10.3b}$$

and the balance equation of f becomes

$$\frac{\partial \rho f}{\partial t} + \nabla_r \cdot \mathbf{J}_{cf} = 0$$

(10.3c)

which means the change rate of \mathcal{F} is due entirely to convective flux.

In general, we can write

$$\rho \frac{df}{dt} = -\nabla \cdot \mathbf{J}_f + \sigma_f$$

(10.3d)

so that (10.3a) takes the form of a **balance equation**

$$\frac{\partial \rho f}{\partial t} + \nabla_r \cdot (\mathbf{J}_{cf} + \mathbf{J}_f) = \sigma_f$$

(10.3e)

where \mathbf{J}_f is the non-convective flux of f and σ_f is called the **source** of f which generates (destroys) f if it is positive (negative).

Reminder: \mathbf{J} is a **flux** if it affects the rate change of a density by the amount $-\nabla_r \cdot \mathbf{J}$.

If $\sigma_f = 0$, any change in f inside a volume is due entirely to the total net flux of f . The corresponding balance equation

$$\frac{\partial \rho f}{\partial t} + \nabla_r \cdot (\mathbf{J}_{cf} + \mathbf{J}_f) = 0 \quad (10.3f)$$

is sometimes called the **equation of continuity** for f .

10.B.1.2. Momentum Balance Equation

The total momentum in volume $V(t)$ is

$$\mathbf{P}(t) = \int_{V(t)} d\mathbf{r} \rho \mathbf{v}$$

(10.4a)

In the presence of an external force per unit mass \mathbf{F} , Newton's law gives

$$\begin{aligned} \frac{d\mathbf{P}}{dt} &= \frac{d}{dt} \int_{V(t)} d\mathbf{r} \rho \mathbf{v} \\ &= \int_{V(t)} d\mathbf{r} \rho \mathbf{F} + \oint_{S(t)} d\mathbf{S} \cdot \mathbb{P} \end{aligned} \quad (10.4b)$$

where \mathbb{P} is the **pressure tensor** acting by the surround fluid on the surface $S(t)$ of $V(t)$. Writing

$$d\mathbf{S} = \hat{\mathbf{n}} dS$$

where $\hat{\mathbf{n}}$ is the (outward) unit vector normal to dS , (10.4b) becomes

$$\frac{d\mathbf{P}}{dt} = \int_{V(t)} d\mathbf{r} \rho \mathbf{F} + \int_S dS \mathbf{f}$$

(10.4)

where

$$\mathbf{f} = \hat{\mathbf{n}} \cdot \mathbb{P}$$

so that $-\mathbf{f}$ is the pressure force (force per unit area) exerted by the surrounding fluid.

Gauss's theorem gives

$$\oint_{S(t)} d\mathbf{S} \cdot \mathbb{P} = \int_{V(t)} d\mathbf{r} \nabla_r \cdot \mathbb{P} \quad (10.5)$$

and (10.4b) becomes

$$\frac{d}{dt} \int_{V(t)} d\mathbf{r} \rho \mathbf{v} = \int_{V(t)} d\mathbf{r} (\rho \mathbf{F} + \nabla_r \cdot \mathbb{P}) \quad (10.6a)$$

On the other hand, similar to (10.1),

$$\frac{d}{dt} \int_{V(t)} d\mathbf{r} \rho \mathbf{v} = \int_{V_z} d\mathbf{z} J \left(\frac{d\rho \mathbf{v}}{dt} + \rho (\nabla_r \cdot \mathbf{v}) \mathbf{v} \right)$$

$$= \int_{V(t)} d\mathbf{r} \left(\frac{d\rho\mathbf{v}}{dt} + \rho(\nabla_r \cdot \mathbf{v})\mathbf{v} \right)$$

so that (10.6a) implies

$$\frac{d\rho\mathbf{v}}{dt} + \rho(\nabla_r \cdot \mathbf{v})\mathbf{v} = \rho\mathbf{F} + \nabla_r \cdot \mathbb{P} \quad (10.6)$$

which is the **momentum balance equation**.

Now, in Cartesian coordinates, $-\mathbb{P}_{ij} = -\mathbb{P}_{ji}$ is the inward force along the j -axis on a surface perpendicular to the i -axis or vice-versa. Therefore, the diagonal elements \mathbb{P}_{ii} are related to pressure, while the off-diagonal elements \mathbb{P}_{ij} denote shear forces caused by viscosity (or friction).

For an **ideal fluid**, there is no viscosity and

$$\mathbb{P} = -P\mathbf{I} \quad (10.6b)$$

where P is the pressure and $\mathbf{I} = \hat{\mathbf{x}}\hat{\mathbf{x}} + \hat{\mathbf{y}}\hat{\mathbf{y}} + \hat{\mathbf{z}}\hat{\mathbf{z}}$ the unit tensor (or matrix).

Reminder: given two vectors \mathbf{a} & \mathbf{b} , the **dyadic** $\mathbf{a}\mathbf{b}$ is a tensor with Cartesian components

$$(\mathbf{a}\mathbf{b})_{ij} = a_i b_j$$

In matrix form with vectors represented by column matrices, we have

$$\mathbf{a}\mathbf{b} = \mathbf{a}\mathbf{b}^T$$

For a non-ideal fluid,

$$\mathbb{P} = -P\mathbf{I} - \mathbb{\Pi}$$

where the **stress tensor** $\mathbb{\Pi}$ denotes dissipative effects caused by collisions between molecules. (10.6) then becomes

$$\frac{d\rho\mathbf{v}}{dt} + \rho(\nabla_r \cdot \mathbf{v})\mathbf{v} = \rho\mathbf{F} - \nabla_r P - \nabla_r \cdot \mathbb{\Pi} \quad (10.7)$$

where we have used

$$\nabla_r \cdot (P\mathbf{I}) = \sum_i \partial_i (P \delta_{ij}) = \partial_j P = \nabla_r P \quad (10.7a)$$

Using

$$\frac{d\rho\mathbf{v}}{dt} = \frac{\partial\rho\mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla_r (\rho\mathbf{v})$$

(10.7) becomes

$$\frac{\partial\rho\mathbf{v}}{\partial t} + \nabla_r \cdot (\rho\mathbf{v}\mathbf{v}) = \rho\mathbf{F} - \nabla_r P - \nabla_r \cdot \mathbb{\Pi}$$

Collecting all ∇_r terms together, we have

$$\frac{\partial\rho\mathbf{v}}{\partial t} + \nabla_r \cdot (\rho\mathbf{v}\mathbf{v} + P\mathbf{I} + \mathbb{\Pi}) = \rho\mathbf{F} \quad (10.8)$$

Either (10.7) or (10.8) can be called **momentum balance equation**. In (10.7), $\frac{d\rho\mathbf{v}}{dt}$ gives the momentum change rate as observed from a coordinate system moving with the fluid, while $\frac{\partial\rho\mathbf{v}}{\partial t}$ in (10.8) describes it from a fixed coordinate system. However, all quantities are functions of \mathbf{r} , which is measured from a fixed coordinate system.

We can use (10.3) to write

$$\begin{aligned}\frac{\partial \rho \mathbf{v}}{\partial t} &= \frac{\partial \rho}{\partial t} \mathbf{v} + \rho \frac{\partial \mathbf{v}}{\partial t} \\ &= -\mathbf{v} \nabla_r \cdot (\rho \mathbf{v}) + \rho \frac{\partial \mathbf{v}}{\partial t}\end{aligned}$$

(10.8) then becomes the **equation of motion** of the fluid

$$\begin{aligned}\rho \frac{\partial \mathbf{v}}{\partial t} &= \mathbf{v} \nabla_r \cdot (\rho \mathbf{v}) - \nabla_r \cdot (\rho \mathbf{v} \mathbf{v} + P \mathbf{I} + \mathbf{\Pi}) + \rho \mathbf{F} \\ &= -\rho (\mathbf{v} \cdot \nabla_r) \mathbf{v} - \nabla_r \cdot (P \mathbf{I} + \mathbf{\Pi}) + \rho \mathbf{F}\end{aligned}$$

(10.8a)

where we have used

$$\begin{aligned}\nabla_r \cdot (\rho \mathbf{v} \mathbf{v}) &= \partial_j (\rho v_j v_i) = v_i \partial_j (\rho v_j) + \rho v_j \partial_j v_i \\ &= \mathbf{v} \nabla_r \cdot (\rho \mathbf{v}) + \rho (\mathbf{v} \cdot \nabla_r) \mathbf{v}\end{aligned}$$

Using (10.7a), (10.8a) becomes

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\rho (\mathbf{v} \cdot \nabla) \mathbf{v} - \nabla_r P - \nabla_r \cdot \mathbf{\Pi} + \rho \mathbf{F} \quad (10.8b)$$

which is called the **Navier-Stokes momentum equation**.

In the absence of viscosity, (10.8a) simplifies to

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla_r) \mathbf{v} = -\frac{1}{\rho} \nabla_r P + \mathbf{F}$$

(10.8b)

which is known as the **Euler equation**.

According to the discussion in §10.B.1.1, (10.8) means that the momentum density $\rho \mathbf{v}$ is not conserved even in the absence of viscosity & external forces, unless $-\nabla_r P = 0$.

In the absence of external forces, (10.8) becomes an equation of continuity for the momentum and describes a generalized form of the conservation of momentum of a fluid particle that includes the effects of the surrounding fluid.

The fluxes $\rho \mathbf{v} \mathbf{v} + P \mathbf{I}$, which do not arise from dissipative processes, are sometimes called **reactive**.

10.B.1.3. Energy & Entropy Balance Equations

The balance equations for energy & entropy are relatively simple in the sense that both are scalar equations. Also, they are not independent but related through thermodynamic relations between entropy and various thermodynamic potentials [see §2.F].

Owing to the close relationship between energy and momentum, we expect the **energy balance equation** to take a form similar to (10.8)

$$\frac{\partial \rho \epsilon}{\partial t} + \nabla_r \cdot (\mathbf{J}_\epsilon^R + \mathbf{J}_\epsilon^D) = \sigma_F$$

(10.9a)

where \mathbf{J}_ϵ^R & \mathbf{J}_ϵ^D are the **reactive** and **dissipative energy flux**, respectively, and σ_F is the **energy source** due to external forces \mathbf{F} . If $\mathbf{F} = 0$, (10.9a) reduces to

$$\frac{\partial \rho \epsilon}{\partial t} + \nabla_r \cdot (\mathbf{J}_\epsilon^R + \mathbf{J}_\epsilon^D) = 0$$

(10.9)

which should also be applicable to conservative forces with $\rho \mathbf{F} = -\nabla_r(\rho \phi)$ since the total energy is “conserved”. In which case, we have

$$\rho \epsilon = \rho u + \frac{1}{2} \rho v^2 + \rho \phi \quad (10.9b)$$

where ϵ and u are the **specific** (or per unit mass) **total** and **internal energies**, respectively.

Similarly, the **entropy balance equation** is

$$\frac{\partial \rho s}{\partial t} + \nabla_r \cdot (\mathbf{J}_s^R + \mathbf{J}_s^D) = \sigma_s$$

(10.10)

where \mathbf{J}_s^R and \mathbf{J}_s^D are the **reactive** and **dissipative entropy flux**, respectively. The **source** term σ_s generates (destroy) entropy if it is positive (negative).

Explicit expressions for \mathbf{J}_ϵ^R , \mathbf{J}_ϵ^D , \mathbf{J}_s^D & σ_s will be derived in the following sections.