10.B.2. Entropy Source and Entropy Current

We now consider a system that varies slowly enough in space and time so that each microscopically large, but macroscopically small, region in it is in (local) thermodynamic equilibrium. In other words, we assume the system can be described by a set of quasi-equilibrium thermodynamic quantities that can be represented as smooth functions of space and time.

Consider the energy density for a system subject to a conservative force density $\boldsymbol{F} = -\nabla_{\boldsymbol{r}} \phi$,

$$\rho \epsilon = \rho u + \frac{1}{2} \rho v^2 + \rho \phi \qquad (10.9a)$$

The kinetic energy density gives

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) &= \rho \boldsymbol{v} \cdot \frac{\partial \boldsymbol{v}}{\partial t} + \frac{1}{2} v^2 \frac{\partial \rho}{\partial t} \\ &= \boldsymbol{v} \cdot \left(\rho \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \frac{\partial \rho}{\partial t}\right) - \frac{1}{2} v^2 \frac{\partial \rho}{\partial t} \\ &= \boldsymbol{v} \cdot \frac{\partial \rho \boldsymbol{v}}{\partial t} - \frac{1}{2} v^2 \frac{\partial \rho}{\partial t} \end{aligned} \qquad (10.9b)$$

Putting in the mass and momentum balance equations (10.3) & (10.8), we have

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = -\boldsymbol{v} \cdot \left[\nabla_{\boldsymbol{r}} \cdot \left(\rho \boldsymbol{v} \boldsymbol{v} + P \boldsymbol{I} + \boldsymbol{\Pi}\right)\right] - \rho \boldsymbol{v} \cdot \nabla_{\boldsymbol{r}} \phi + \frac{1}{2} v^2 \nabla_{\boldsymbol{r}} \cdot (\rho \boldsymbol{v}) \qquad (10.14a)$$

Using

$$\begin{aligned} \boldsymbol{v} \cdot \nabla_{\boldsymbol{r}} \cdot (\rho \boldsymbol{v} \boldsymbol{v}) &= v_i \partial_j (\rho v_j v_i) = v_i \left[v_i \partial_j (\rho v_j) + \rho v_j \partial_j v_i \right] \\ &= v^2 \nabla_{\boldsymbol{r}} \cdot (\rho \boldsymbol{v}) + \frac{1}{2} \rho \boldsymbol{v} \cdot \nabla_{\boldsymbol{r}} v^2 \end{aligned}$$

$$\nabla_{\boldsymbol{r}} \cdot \left(\rho v^2 \boldsymbol{v}\right) = v^2 \nabla_{\boldsymbol{r}} \cdot (\rho \boldsymbol{v}) + \rho \boldsymbol{v} \cdot \nabla_{\boldsymbol{r}} v^2$$

we have

$$\boldsymbol{v} \cdot \nabla_{\boldsymbol{r}} \cdot (\rho \boldsymbol{v} \boldsymbol{v}) = \frac{1}{2} \nabla_{\boldsymbol{r}} \cdot \left(\rho v^2 \boldsymbol{v}\right) + \frac{1}{2} v^2 \nabla_{\boldsymbol{r}} \cdot (\rho \boldsymbol{v}) \qquad (10.19)$$

Also, for any tensor $\boldsymbol{G}$,

$$\begin{aligned} \boldsymbol{v} \cdot (\nabla_{\boldsymbol{r}} \cdot \boldsymbol{G}) &= v_i \partial_j G_{ji} = \partial_j (v_i G_{ji}) - G_{ji} \partial_j v_i \\ &= \nabla_{\boldsymbol{r}} \cdot (\boldsymbol{G} \cdot \boldsymbol{v}) - \left(\boldsymbol{G}^T \cdot \nabla_{\boldsymbol{r}}\right) \cdot \boldsymbol{v} \end{aligned} \qquad (10.19a)$$

Thus,

$$\boldsymbol{v} \cdot \left[\nabla_{\boldsymbol{r}} \cdot (P \boldsymbol{I} + \boldsymbol{\Pi}) \right] = \nabla_{\boldsymbol{r}} \cdot (P \boldsymbol{v} + \boldsymbol{\Pi} \cdot \boldsymbol{v}) - \left[\left(P \boldsymbol{I} + \boldsymbol{\Pi}^T\right) \cdot \nabla_{\boldsymbol{r}} \right] \cdot \boldsymbol{v}$$

which, with

$$(P \boldsymbol{I} \cdot \nabla_{\boldsymbol{r}}) \cdot \boldsymbol{v} = P \delta_{ij} \partial_j v_i = P \partial_i v_i = P \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v}$$

becomes

$$\boldsymbol{v} \cdot \left[\nabla_{\boldsymbol{r}} \cdot (P \boldsymbol{I} + \boldsymbol{\Pi}) \right] = \nabla_{\boldsymbol{r}} \cdot (P \boldsymbol{v} + \boldsymbol{\Pi} \cdot \boldsymbol{v}) - P \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v} - \left(\boldsymbol{\Pi}^T \cdot \nabla_{\boldsymbol{r}}\right) \cdot \boldsymbol{v}$$

Combining with (10.19) then gives

$$\begin{aligned} &\boldsymbol{v} \cdot \left[\nabla_{\boldsymbol{r}} \cdot \left(\rho \boldsymbol{v} \boldsymbol{v} + P \boldsymbol{I} + \boldsymbol{\Pi}\right)\right] \\ &= \nabla_{\boldsymbol{r}} \cdot \left[\left(\frac{1}{2} \rho v^2 + P\right) \boldsymbol{v} + \boldsymbol{\Pi} \cdot \boldsymbol{v} \right] + \frac{1}{2} v^2 \nabla_{\boldsymbol{r}} \cdot (\rho \boldsymbol{v}) - P \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v} - \left(\boldsymbol{\Pi}^T \cdot \nabla_{\boldsymbol{r}}\right) \cdot \boldsymbol{v} \end{aligned} \qquad (10.19b)$$

(10.14a) thus becomes

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = -\nabla_{\boldsymbol{r}} \cdot \left[\left(\frac{1}{2} \rho v^2 + P\right) \boldsymbol{v} + \boldsymbol{\Pi} \cdot \boldsymbol{v} \right] + P \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v} + \left(\boldsymbol{\Pi}^T \cdot \nabla_{\boldsymbol{r}}\right) \cdot \boldsymbol{v} - \rho \boldsymbol{v} \cdot \nabla_{\boldsymbol{r}} \phi$$

$$= -\nabla_{\boldsymbol{r}} \cdot \left[\rho \left(\frac{1}{2} v^2 + \frac{P}{\rho} + \phi \right) \boldsymbol{v} + \boldsymbol{\Pi} \cdot \boldsymbol{v} \right] + P \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v} + \left(\boldsymbol{\Pi}^T \cdot \nabla_{\boldsymbol{r}} \right) \cdot \boldsymbol{v} + \phi \nabla_{\boldsymbol{r}} \cdot (\rho \boldsymbol{v})$$

where we have used

$$\nabla_{\boldsymbol{r}} \cdot (\rho \phi \boldsymbol{v}) = \rho \boldsymbol{v} \cdot \nabla_{\boldsymbol{r}} \phi + \phi \nabla_{\boldsymbol{r}} \cdot (\rho \boldsymbol{v})$$

Using the mass balance equation, we get

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \rho v^2 \right) + \frac{\partial \rho}{\partial t} \phi = -\nabla_{\boldsymbol{r}} \cdot \left[\rho \left(\frac{1}{2} v^2 + \frac{P}{\rho} + \phi \right) \boldsymbol{v} + \boldsymbol{\Pi} \cdot \boldsymbol{v} \right] + P \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v} + \left(\boldsymbol{\Pi}^T \cdot \nabla_{\boldsymbol{r}} \right) \cdot \boldsymbol{v} \qquad (10.19b)$$

$$= -\nabla_{\boldsymbol{r}} \cdot \left[\rho \left(\frac{1}{2} v^2 + \phi \right) \boldsymbol{v} + \boldsymbol{\Pi} \cdot \boldsymbol{v} \right] - \boldsymbol{v} \cdot \nabla_{\boldsymbol{r}} P + \left(\boldsymbol{\Pi}^T \cdot \nabla_{\boldsymbol{r}} \right) \cdot \boldsymbol{v} \qquad (10.19c)$$

where

$$\nabla_{\boldsymbol{r}} \cdot (P \boldsymbol{v}) = \boldsymbol{v} \cdot \nabla_{\boldsymbol{r}} P + P \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v}$$

was used.

Taking the time derivative of (10.9a) gives

$$\frac{\partial (\rho \epsilon)}{\partial t} = \frac{\partial (\rho u)}{\partial t} + \frac{\partial}{\partial t} \left(\frac{1}{2} \rho v^2 \right) + \frac{\partial \rho}{\partial t} \phi$$

$$= \frac{\partial (\rho u)}{\partial t} + \boldsymbol{v} \cdot \frac{\partial \rho \boldsymbol{v}}{\partial t} + \left(\phi - \frac{1}{2} v^2 \right) \frac{\partial \rho}{\partial t} \quad \text{[(10.9b) used.]} \qquad (10.14)$$

On the other hand, using (10.19b) instead gives

$$\frac{\partial (\rho \epsilon)}{\partial t} = \frac{\partial (\rho u)}{\partial t} - \nabla_{\boldsymbol{r}} \cdot \left[\rho \left(\frac{1}{2} v^2 + \frac{P}{\rho} + \phi \right) \boldsymbol{v} + \boldsymbol{\Pi} \cdot \boldsymbol{v} \right] + P \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v} + \left(\boldsymbol{\Pi}^T \cdot \nabla_{\boldsymbol{r}} \right) \cdot \boldsymbol{v} \qquad (10.19d)$$

$$= \frac{\partial (\rho u)}{\partial t} - \nabla_{\boldsymbol{r}} \cdot \left[\rho \left(\frac{1}{2} v^2 + \phi \right) \boldsymbol{v} + \boldsymbol{\Pi} \cdot \boldsymbol{v} \right] - \boldsymbol{v} \cdot \nabla_{\boldsymbol{r}} P + \left(\boldsymbol{\Pi}^T \cdot \nabla_{\boldsymbol{r}} \right) \cdot \boldsymbol{v} \qquad (10.19e)$$

The $\frac{\partial (\rho u)}{\partial t}$ term can be replaced by means of thermodynamical relations that are derived as follows.

Consider the fundamental equation for the internal energy [see (2.66)]

$$U = T S - P V + \mu N$$

where μ is the chemical potential per particle and N is the number of particles in V.

Dividing by V gives

$$\frac{U}{V} = T \frac{S}{V} - P + \mu \frac{N}{V}$$

$$\rightarrow \quad \rho \frac{U}{M} = \rho T \frac{S}{M} - P + \rho \frac{\mu}{M} N \qquad \rho = \frac{M}{V} = \text{mass density}$$

$$\rho u = T \rho s - P + \rho \tilde{\mu} \qquad \tilde{\mu} = \frac{\mu}{M / N} = \frac{\mu}{m} \qquad (10.11a)$$

where M is the mass of the fluid in volume V and m the mass of each particle. u, s and $\tilde{\mu}$ are the specific (or per unit mass) energy, entropy and chemical potential, respectively.

Taking the derivative of (10.11a) gives

$$d(\rho u) = T\, d(\rho s) + \rho s\, d T - d P + \rho\, d \tilde{\mu} + \tilde{\mu}\, d \rho$$

Subtracting from this the Gibbs-Duhem equation [see (2.62)]

$$N\, d \mu + S\, d T - V\, d P = 0$$

or

$$\rho\, d \tilde{\mu} + \rho s\, d T - d P = 0 \qquad (10.11b)$$

we get

$$d(\rho u) = T\, d(\rho s) + \tilde{\mu}\, d \rho \qquad (10.11c)$$

Dividing by $d\,t$ gives

$$0 = \frac{d(\rho\,u)}{d\,t} - T\,\frac{d(\rho\,s)}{d\,t} - \tilde{\mu}\,\frac{d\,\rho}{d\,t} \qquad (10.11)$$

$$= \frac{\partial(\rho\,u)}{\partial\,t} + \boldsymbol{v}\cdot\nabla_{\boldsymbol{r}}(\rho\,u) - T\,\frac{\partial(\rho\,s)}{\partial\,t} - T\,\boldsymbol{v}\cdot\nabla_{\boldsymbol{r}}(\rho\,s) - \tilde{\mu}\,\frac{\partial\,\rho}{\partial\,t} - \tilde{\mu}\,\boldsymbol{v}\cdot\nabla_{\boldsymbol{r}}\rho$$

where (10.2a) was used.

Now,

$$\begin{aligned}
&\boldsymbol{v}\cdot\nabla_{\boldsymbol{r}}(\rho\,u) - T\,\boldsymbol{v}\cdot\nabla_{\boldsymbol{r}}(\rho\,s) - \tilde{\mu}\,\boldsymbol{v}\cdot\nabla_{\boldsymbol{r}}\rho \\
&= \boldsymbol{v}\cdot\nabla_{\boldsymbol{r}}(\rho\,u - T\,\rho\,s - \tilde{\mu}\,\rho) + \rho\,s\,\boldsymbol{v}\cdot\nabla_{\boldsymbol{r}}T + \rho\,\boldsymbol{v}\cdot\nabla_{\boldsymbol{r}}\tilde{\mu} \\
&= -\boldsymbol{v}\cdot\nabla_{\boldsymbol{r}}P + \rho\,s\,\boldsymbol{v}\cdot\nabla_{\boldsymbol{r}}T + \rho\,\boldsymbol{v}\cdot\nabla_{\boldsymbol{r}}\tilde{\mu} \qquad \text{[(10.11a) used.]} \\
&= -\boldsymbol{v}\cdot(\nabla_{\boldsymbol{r}}P - \rho\,s\,\nabla_{\boldsymbol{r}}T - \rho\,\nabla_{\boldsymbol{r}}\tilde{\mu}) \\
&= 0 \qquad \text{[(10.11b) used with } d\,f \to \boldsymbol{r}\cdot\nabla_r f \text{]} \qquad (10.12)
\end{aligned}$$

(10.11) thus simplifies to

$$\frac{\partial(\rho\,u)}{\partial\,t} - T\,\frac{\partial(\rho\,s)}{\partial\,t} - \tilde{\mu}\,\frac{\partial\,\rho}{\partial\,t} = 0 \qquad (10.13)$$

Using (10.13) to replace $\frac{\partial(\rho\,u)}{\partial\,t}$ in (10.14) then gives

$$T\,\frac{\partial(\rho\,s)}{\partial\,t} = \frac{\partial(\rho\,\epsilon)}{\partial\,t} - \boldsymbol{v}\cdot\frac{\partial\rho\,\boldsymbol{v}}{\partial\,t} + \left(\frac{1}{2}\,v^2 - \tilde{\mu} - \phi\right)\frac{\partial\,\rho}{\partial\,t} \qquad (10.15)$$

Putting in the mass and momentum balance equations (10.3) & (10.8), we can write (10.15) as

$$T\,\frac{\partial(\rho\,s)}{\partial\,t} = -\nabla_{\boldsymbol{r}}\cdot\left(\boldsymbol{J}_\epsilon^R + \boldsymbol{J}_\epsilon^D\right) + \boldsymbol{v}\cdot\left[\nabla_{\boldsymbol{r}}\cdot\left(\rho\,\boldsymbol{v}\,\boldsymbol{v} + P\,\boldsymbol{I} + \boldsymbol{\Pi}\right)\right] \qquad (10.16)$$

$$+\rho\,\boldsymbol{v}\cdot\nabla_r\phi - \left(\frac{1}{2}\,v^2 - \tilde{\mu} - \phi\right)\nabla_{\boldsymbol{r}}\cdot(\rho\,\boldsymbol{v})$$

(10.16) becomes

$$T\,\frac{\partial(\rho\,s)}{\partial\,t} = -\nabla_{\boldsymbol{r}}\cdot\left(\boldsymbol{J}_\epsilon^R + \boldsymbol{J}_\epsilon^D\right) + \nabla_{\boldsymbol{r}}\cdot\left[\left(\frac{1}{2}\,\rho\,v^2 + P\right)\boldsymbol{v} + \boldsymbol{\Pi}\cdot\boldsymbol{v}\right] + \rho\,\boldsymbol{v}\cdot\nabla_{\boldsymbol{r}}\phi \qquad (10.19b)$$

$$+(\tilde{\mu} + \phi)\,\nabla_{\boldsymbol{r}}\cdot(\rho\,\boldsymbol{v}) - P\,\nabla_{\boldsymbol{r}}\cdot\boldsymbol{v} - \left(\boldsymbol{\Pi}^T\cdot\nabla_{\boldsymbol{r}}\right)\cdot\boldsymbol{v}$$

Using

$$\nabla_{\boldsymbol{r}}\cdot\left[\left(\tilde{\mu} + \phi - \frac{P}{\rho}\right)\rho\,\boldsymbol{v}\right] = (\tilde{\mu} + \phi)\,\nabla_{\boldsymbol{r}}\cdot(\rho\,\boldsymbol{v}) - P\,\nabla_{\boldsymbol{r}}\cdot\boldsymbol{v} + \rho\,\boldsymbol{v}\cdot\nabla_{\boldsymbol{r}}(\tilde{\mu} + \phi) - \boldsymbol{v}\cdot\nabla_{\boldsymbol{r}}P$$

(10.19b) becomes

$$T\,\frac{\partial(\rho\,s)}{\partial\,t} = -\nabla_{\boldsymbol{r}}\cdot\left(\boldsymbol{J}_\epsilon^R + \boldsymbol{J}_\epsilon^D\right) + \nabla_{\boldsymbol{r}}\cdot\left[\rho\left(\frac{1}{2}\,v^2 + \tilde{\mu} + \phi\right)\boldsymbol{v} + \boldsymbol{\Pi}\cdot\boldsymbol{v}\right] \qquad (10.19c)$$

$$-\rho\,\boldsymbol{v}\cdot\nabla_r\tilde{\mu} + \boldsymbol{v}\cdot\nabla_r P - \left(\boldsymbol{\Pi}^T\cdot\nabla_{\boldsymbol{r}}\right)\cdot\boldsymbol{v}$$

Replacing d with $\boldsymbol{v}\cdot\nabla_r$ in the Gibbs-Duhem equation (10.11b) gives

$$\rho\,\boldsymbol{v}\cdot\nabla_r\tilde{\mu} + \rho\,s\,\boldsymbol{v}\cdot\nabla_{\boldsymbol{r}}\,T - \boldsymbol{v}\cdot\nabla_{\boldsymbol{r}}\,P = 0$$

which turns (10.19c) into

$$T\,\frac{\partial(\rho\,s)}{\partial\,t} = -\nabla_{\boldsymbol{r}}\cdot\left(\boldsymbol{J}_\epsilon^R + \boldsymbol{J}_\epsilon^D - \rho\,\tilde{\mu}'\,\boldsymbol{v} - \boldsymbol{\Pi}\cdot\boldsymbol{v}\right) + \rho\,s\,\boldsymbol{v}\cdot\nabla_{\boldsymbol{r}}\,T - \left(\boldsymbol{\Pi}^T\cdot\nabla_{\boldsymbol{r}}\right)\cdot\boldsymbol{v} \qquad (10.20)$$

where

$$\tilde{\mu}' = \frac{1}{2}\,v^2 + \tilde{\mu} + \phi \qquad (10.20a)$$

Using

$$\nabla_r \cdot \left(\frac{\boldsymbol{A}}{T} \right) = \frac{1}{T} \nabla_r \cdot \boldsymbol{A} + \boldsymbol{A} \cdot \nabla_r \frac{1}{T}$$

and

$$\boldsymbol{v} \cdot \nabla_r T = -T^2 \boldsymbol{v} \cdot \nabla_r \frac{1}{T}$$

(10.20) can be put into the form of a balance equation:

$$\frac{\partial(\rho s)}{\partial t} + \nabla_r \cdot \left(\frac{\boldsymbol{J}_\epsilon^R + \boldsymbol{J}_\epsilon^D - \rho \tilde{\mu}' \boldsymbol{v} - \boldsymbol{\Pi} \cdot \boldsymbol{v}}{T} \right) \tag{10.21}$$
$$= \left[\boldsymbol{J}_\epsilon^R + \boldsymbol{J}_\epsilon^D - \rho (\tilde{\mu}' + s T) \boldsymbol{v} - \boldsymbol{\Pi} \cdot \boldsymbol{v} \right] \cdot \nabla_r \frac{1}{T} - \frac{1}{T} \left(\boldsymbol{\Pi}^T \cdot \nabla_r \right) \cdot \boldsymbol{v}$$

Consider now a fluid with no dissipation and hence no entropy source, i.e.,

$$\boldsymbol{\Pi} = 0 \qquad \boldsymbol{J}_\epsilon^D = 0 \qquad \& \qquad \boldsymbol{J}_\epsilon^R + \boldsymbol{J}_\epsilon^D - \rho (\tilde{\mu}' + s T) \boldsymbol{v} - \boldsymbol{\Pi} \cdot \boldsymbol{v} = 0$$

$$\rightarrow \qquad \boldsymbol{J}_\epsilon^R = \rho (\tilde{\mu}' + s T) \boldsymbol{v}$$
$$= \rho \left(\frac{1}{2} v^2 + \phi + u + \frac{P}{\rho} \right) \boldsymbol{v} \qquad \text{[(10.11a) used.]} \tag{10.22}$$
$$= \rho \left(\frac{1}{2} v^2 + \phi + h \right) \boldsymbol{v} \qquad h = u + \frac{P}{\rho} = \text{specific enthalpy}.$$
$$= \rho \left(\epsilon + \frac{P}{\rho} \right) \boldsymbol{v}$$
$$= \boldsymbol{J}_\epsilon^c + P \boldsymbol{v} \qquad \boldsymbol{J}_\epsilon^c = \rho \epsilon \boldsymbol{v} = \text{convective energy flux}.$$

(10.21) then becomes

$$\frac{\partial(\rho s)}{\partial t} + \nabla_r \cdot (\rho s \boldsymbol{v}) = 0 \tag{10.23}$$

which is the entropy balance equation for non-dissipative fluids. Note that (10.23) is also an equation of continuity so that s is conserved [see §10.B.1.1.1]:

$$\frac{d s}{d t} = 0$$

i.e., the flow is adiabatic.

Since $\boldsymbol{J}_\epsilon^R$ is the same irregardless of the presence of dissipation, we can put (10.22) back into (10.21) to get

$$\frac{\partial(\rho s)}{\partial t} + \nabla_r \cdot \left(\frac{\rho s T \boldsymbol{v} + \boldsymbol{J}_\epsilon^D - \boldsymbol{\Pi} \cdot \boldsymbol{v}}{T} \right) = \left(\boldsymbol{J}_\epsilon^D - \boldsymbol{\Pi} \cdot \boldsymbol{v} \right) \cdot \nabla_r \frac{1}{T} - \frac{1}{T} \left(\boldsymbol{\Pi}^T \cdot \nabla_r \right) \cdot \boldsymbol{v} \tag{10.24a}$$

Setting the dissipative entropy flow as

$$\boldsymbol{J}_s^D = \frac{\boldsymbol{J}_\epsilon^D - \boldsymbol{\Pi} \cdot \boldsymbol{v}}{T} \tag{10.25}$$

we have the final form of the entropy balance equation

$$\frac{\partial(\rho s)}{\partial t} + \nabla_r \cdot \left(\rho s \boldsymbol{v} + \boldsymbol{J}_s^D \right) = -\frac{1}{T} \boldsymbol{J}_s^D \cdot \nabla_r T - \frac{1}{T} \left(\boldsymbol{\Pi}^T \cdot \nabla_r \right) \cdot \boldsymbol{v} \tag{10.24}$$

from which we can identify the entropy source as

$$\sigma_s = -\frac{1}{T} \boldsymbol{J}_s^D \cdot \nabla_r T - \frac{1}{T} \left(\boldsymbol{\Pi}^T \cdot \nabla_r \right) \cdot \boldsymbol{v} \tag{10.26}$$

Exercise 10.1

(a) Write the dyadic tensor $\nabla_{\boldsymbol{r}} \boldsymbol{v}$ in orthogonal curvilinear coordinates.
Assume coordinates u_1, u_2, u_3, unit vectors $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ & scale factors h_1, h_2, h_3.

(b) Write $\nabla_{\boldsymbol{r}} \boldsymbol{v}$ in (circular) cylindrical coordinates with

$$(u_1, u_2, u_3) = (r, \phi, z) \qquad (\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3) = \left(\hat{\boldsymbol{r}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{z}}\right) \quad (h_1, h_2, h_3) = (1, r, 1)$$

Answer (a)

For orthogonal curvilinear coordinates,

$$\nabla_{\boldsymbol{r}} = \hat{\mathbf{e}}_1 \frac{1}{h_1} \frac{\partial}{\partial u_1} + \hat{\mathbf{e}}_2 \frac{1}{h_2} \frac{\partial}{\partial u_2} + \hat{\mathbf{e}}_3 \frac{1}{h_3} \frac{\partial}{\partial u_3} \tag{1}$$

$$= \hat{\boldsymbol{u}}_j \frac{\partial}{\partial u_j} \qquad \text{(Summation over pair of repeated indices implied.)}$$

where $\hat{\boldsymbol{u}}_j = \hat{\mathbf{e}}_j \frac{1}{h_j}$ (no summation over j).

$$\boldsymbol{v} = v_{u\,1} \hat{\mathbf{e}}_1 + v_{u\,2} \hat{\mathbf{e}}_2 + v_{u\,3} \hat{\mathbf{e}}_3 = v_{u\,j} \hat{\mathbf{e}}_j$$

For the special case of Cartesian coordinates

$$\nabla_{\boldsymbol{r}} = \hat{\boldsymbol{x}}_1 \frac{\partial}{\partial x_1} + \hat{\boldsymbol{x}}_2 \frac{\partial}{\partial x_2} + \hat{\boldsymbol{x}}_3 \frac{\partial}{\partial x_3} = \hat{\boldsymbol{x}}_\alpha \frac{\partial}{\partial x_\alpha} \qquad \alpha = 1, 2, 3$$

$$\boldsymbol{v} = v_{x\,1} \hat{\boldsymbol{x}}_1 + v_{x\,2} \hat{\boldsymbol{x}}_2 + v_{x\,3} \hat{\boldsymbol{x}}_3 = v_{x\,\alpha} \hat{\boldsymbol{x}}_\alpha$$

The unit vectors of different orthogonal coordinates are related by orthogonal transforms.

Usually, one begins by defining the curvilinear coordinates in terms of the Cartesian ones

$$u_j = u_j(x_\alpha) \qquad j = 1, 2, 3$$

so that for any vector $\boldsymbol{v}$,

$$\boldsymbol{v} = v_{x\,1} \hat{\boldsymbol{x}}_1 + v_{x\,2} \hat{\boldsymbol{x}}_2 + v_{x\,3} \hat{\boldsymbol{x}}_3 = v_{u\,1} \hat{\mathbf{e}}_1 + v_{u\,2} \hat{\mathbf{e}}_2 + v_{u\,3} \hat{\mathbf{e}}_3$$

$$= v_{x\,\alpha} \hat{\boldsymbol{x}}_\alpha = v_{u\,j} \hat{\mathbf{e}}_j$$

and

$$\hat{\mathbf{e}}_j = R_{j\,\alpha} \hat{\boldsymbol{x}}_\alpha \qquad \hat{\boldsymbol{x}}_\alpha = R^{-1}_{\alpha j} \hat{\mathbf{e}}_j = R_{j\,\alpha} \hat{\mathbf{e}}_j \tag{2}$$

The dyadic tensor $\nabla_{\boldsymbol{r}} \boldsymbol{v}$ is defined as

$$\nabla_{\boldsymbol{r}} \boldsymbol{v} = \left(\hat{\mathbf{e}}_1 \frac{1}{h_1} \frac{\partial}{\partial u_1} + \hat{\mathbf{e}}_2 \frac{1}{h_2} \frac{\partial}{\partial u_2} + \hat{\mathbf{e}}_3 \frac{1}{h_3} \frac{\partial}{\partial u_3}\right) \left(v_{u\,1} \hat{\mathbf{e}}_1 + v_{u\,2} \hat{\mathbf{e}}_2 + v_{u\,3} \hat{\mathbf{e}}_3\right)$$

$$= \hat{\boldsymbol{u}}_j \frac{\partial}{\partial u_j} \left(v_{u\,k} \hat{\mathbf{e}}_k\right)$$

$$= \hat{\boldsymbol{u}}_j \left(\frac{\partial v_{u\,k}}{\partial u_j} \hat{\mathbf{e}}_k + v_{u\,k} \frac{\partial \hat{\mathbf{e}}_k}{\partial u_j}\right) \tag{2a}$$

To evaluate $\frac{\partial \hat{\mathbf{e}}_k}{\partial u_j}$, we use (2) to get

$$\frac{\partial \hat{\mathbf{e}}_k}{\partial u_j} = \frac{\partial}{\partial u_j} \left(R_{k\,\alpha} \hat{\boldsymbol{x}}_\alpha\right) = \hat{\boldsymbol{x}}_\alpha \frac{\partial R_{k\,\alpha}}{\partial u_j} = R^{-1}_{\alpha\,m} \hat{\mathbf{e}}_m \frac{\partial R_{k\,\alpha}}{\partial u_j}$$

(2a) then becomes

$$\nabla_{\boldsymbol{r}} \boldsymbol{v} = \hat{\boldsymbol{u}}_j \left(\hat{\mathbf{e}}_k \frac{\partial v_{u\,k}}{\partial u_j} + \hat{\mathbf{e}}_m v_{u\,k} \frac{\partial R_{k\,\alpha}}{\partial u_j} R^{-1}_{\alpha\,m}\right)$$

$$= \sum_{j,k} \frac{\hat{\mathbf{e}}_j}{h_j} \left(\hat{\mathbf{e}}_k \frac{\partial v_{u\,k}}{\partial u_j} + \sum_{m,\alpha} \hat{\mathbf{e}}_m v_{u\,k} \frac{\partial R_{k\,\alpha}}{\partial u_j} R^{-1}_{\alpha\,m} \right) \quad \text{[No implicit summation.]} \qquad (3)$$

$$= \hat{\mathbf{u}}_j \hat{\mathbf{e}}_k \left(\frac{\partial v_{u\,k}}{\partial u_j} + v_{u\,m} \frac{\partial R_{m\,\alpha}}{\partial u_j} R^{-1}_{\alpha\,k} \right)$$

$$= \sum_{j,k} \frac{\hat{\mathbf{e}}_j}{h_j} \hat{\mathbf{e}}_k \left(\frac{\partial v_{u\,k}}{\partial u_j} + \sum_{m,\alpha} v_{u\,m} \frac{\partial R_{m\,\alpha}}{\partial u_j} R^{-1}_{\alpha\,k} \right) \qquad \text{(N.I.S.)}$$

with components

$$(\nabla_r \mathbf{v})_{j\,k} = \frac{1}{h_j} \left(\frac{\partial v_{u\,k}}{\partial u_j} + \sum_{m,\alpha} v_{u\,m} \frac{\partial R_{m\,\alpha}}{\partial u_j} R^{-1}_{\alpha\,k} \right) \qquad \text{(N.I.S.)} \qquad (3a)$$

In the special case of the Cartesian coordinates,

$$\nabla_r \mathbf{v} = \hat{\mathbf{x}}_\alpha \frac{\partial}{\partial x_\alpha} (v_{x\,\beta} \hat{\mathbf{x}}_\beta)$$

$$= \hat{\mathbf{x}}_\alpha \hat{\mathbf{x}}_\beta \frac{\partial v_{x\,\beta}}{\partial x_\alpha} \qquad (3b)$$

with components

$$(\nabla_r \mathbf{v})_{\alpha\,\beta} = \frac{\partial v_{x\,\beta}}{\partial x_\alpha} \qquad (3c)$$

Answer (b)

In cylindrical coordinates, simple geometry shows that

$$\begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}}_1 \\ \hat{\mathbf{x}}_2 \\ \hat{\mathbf{x}}_3 \end{pmatrix} \qquad (4)$$

$$\rightarrow \qquad R = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad R^{-1} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

With the help of the *Mathematica* code in §Code, we get

$$\nabla_r \mathbf{v} = \begin{pmatrix} \partial_r v_r & \partial_r v_\phi & \partial_r v_z \\ \frac{1}{r}(-v_\phi + \partial_\phi v_r) & \frac{1}{r}(v_r + \partial_\phi v_\phi) & \frac{1}{r}\partial_\phi v_z \\ \partial_z v_r & \partial_z v_\phi & \partial_z v_z \end{pmatrix}$$

$$= \hat{\mathbf{r}}\hat{\mathbf{r}} \partial_r v_r + \hat{\mathbf{r}}\hat{\boldsymbol{\phi}} \partial_r v_\phi + \hat{\mathbf{r}}\hat{\mathbf{z}} \partial_r v_z \qquad (5)$$

$$+ \hat{\boldsymbol{\phi}}\hat{\mathbf{r}} \frac{1}{r}(-v_\phi + \partial_\phi v_r) + \hat{\boldsymbol{\phi}}\hat{\boldsymbol{\phi}} \frac{1}{r}(v_r + \partial_\phi v_\phi) + \hat{\boldsymbol{\phi}}\hat{\mathbf{z}} \frac{1}{r}\partial_\phi v_z$$

$$+ \hat{\mathbf{z}}\hat{\mathbf{r}} \partial_z v_r + \hat{\mathbf{z}}\hat{\boldsymbol{\phi}} \partial_z v_\phi + \hat{\mathbf{z}}\hat{\mathbf{z}} \partial_z v_z$$

Code

```
In[7]:= u = {r, ϕ, z}; h = {1, r, 1};
        v = {vr[u], vϕ[u], vz[u]};

In[10]:= R := ( Cos[ϕ]   Sin[ϕ]   0
               -Sin[ϕ]   Cos[ϕ]   0
                 0         0      1 );
```

$$(\nabla_r \boldsymbol{v})_{jk} = \frac{1}{h_j}\left(\frac{\partial v_{uk}}{\partial u_j} + \sum_{m,\alpha} v_{um} \frac{\partial R_{m\alpha}}{\partial u_j} R^{-1}_{\alpha k}\right)$$

In[13]:=
```
delv[j_, k_] := 1/h[[j]] (∂_{u[[j]]} v[[k]] + Sum[Sum[v[[m]] (∂_{u[[j]]} R[[m, α]]) R[[k, α]], {α, 1, 3}], {m, 1, 3}])
```

In[16]:=
```
Table[delv[j, k], {j, 3}, {k, 3}] // Simplify // TableForm
```

Out[16]//TableForm=

$vr^{(\{1,0,0\})}[\{r,\phi,z\}]$	$v\phi^{(\{1,0,0\})}[\{r,\phi,z\}]$	$vz^{(\{1,0,0\})}[\{r,\phi,z\}]$
$\frac{-v\phi[\{r,\phi,z\}]+vr^{(\{0,1,0\})}[\{r,\phi,z\}]}{r}$	$\frac{vr[\{r,\phi,z\}]+v\phi^{(\{0,1,0\})}[\{r,\phi,z\}]}{r}$	$\frac{vz^{(\{0,1,0\})}[\{r,\phi,z\}]}{r}$
$vr^{(\{0,0,1\})}[\{r,\phi,z\}]$	$v\phi^{(\{0,0,1\})}[\{r,\phi,z\}]$	$vz^{(\{0,0,1\})}[\{r,\phi,z\}]$