10.B.3. Transport Coefficients

The prototype of transport coefficients is the (electrical) conductivity σ linking the (electrical) current density $\boldsymbol{J}$ to the applied electric field $\boldsymbol{E}$ by

$$\boldsymbol{J} = \sigma \boldsymbol{E} = -\sigma \nabla_{\boldsymbol{r}} \phi$$

where ϕ is the (electrical) potential. This can be generalized in two ways. First, we can replace ϕ with other fields such as T, $\tilde{\mu}$, or even the vector $\boldsymbol{v}$. Secondly, we can generalize σ to a tensor, which is necessary for an anisotropic system.

In the case of the tensor driving force $-\nabla_{\boldsymbol{r}} \boldsymbol{v}$, the corresponding current density is the stress tensor $\boldsymbol{\Pi}$. This comes about as follows.

From the molecular point of view, dissipation (or viscosity) can be caused only by molecular scattering, which in turn requires neighboring molecules to travel with different velocities, i.e., $\nabla_{\boldsymbol{r}} \boldsymbol{v} \neq 0$. Thus,

$$\boldsymbol{\Pi} \propto -\nabla_{\boldsymbol{r}} \boldsymbol{v} \tag{1a}$$

Furthermore, there is also no scattering if all molecules are rotating with the same constant angular velocity $\boldsymbol{\Omega}$. In which case,

$$\boldsymbol{v} = \boldsymbol{\Omega} \times \boldsymbol{r}$$

$$\rightarrow \quad (\nabla_{\boldsymbol{r}} \boldsymbol{v})_{jk} = \partial_j v_k = \partial_j (\epsilon_{kmn} \Omega_m x_n) = \epsilon_{kmn} \Omega_m \delta_{jn} = \epsilon_{kmj} \Omega_m \neq 0$$

so that (1a) must be modified. With $j \leftrightarrow k$, we get

$$(\nabla_{\boldsymbol{r}} \boldsymbol{v})_{kj} = \epsilon_{jmk} \Omega_m = -\epsilon_{kmj} \Omega_m$$

$$\rightarrow \quad (\nabla_{\boldsymbol{r}} \boldsymbol{v})_{jk} + (\nabla_{\boldsymbol{r}} \boldsymbol{v})_{kj} = 0 \qquad \text{if} \qquad \boldsymbol{v} = \boldsymbol{\Omega} \times \boldsymbol{r} \tag{1b}$$

(1a) is thus modified to read

$$\boldsymbol{\Pi} \propto -\left[\nabla_{\boldsymbol{r}} \boldsymbol{v} + (\nabla_{\boldsymbol{r}} \boldsymbol{v})^T \right] \tag{1c}$$

so that $\boldsymbol{\Pi}$ is symmetric.

As shown in §(Useful Mathematics)], $\nabla_{\boldsymbol{r}} \boldsymbol{v}$ can be decomposed into 3 orthogonal components

$$\nabla_{\boldsymbol{r}} \boldsymbol{v} = \frac{1}{3} \mathrm{Tr}(\nabla_{\boldsymbol{r}} \boldsymbol{v})\, \mathbb{I} + (\nabla_{\boldsymbol{r}} \boldsymbol{v})^s + (\nabla_{\boldsymbol{r}} \boldsymbol{v})^a \tag{10.27}$$

where

$$\mathrm{Tr}(\nabla_{\boldsymbol{r}} \boldsymbol{v}) = \partial_j v_j = \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v}$$

$$(\nabla_{\boldsymbol{r}} \boldsymbol{v})^s = \frac{1}{2}\left[\nabla_{\boldsymbol{r}} \boldsymbol{v} + (\nabla_{\boldsymbol{r}} \boldsymbol{v})^T \right] - \frac{1}{3} \mathrm{Tr}(\nabla_{\boldsymbol{r}} \boldsymbol{v})\, \mathbb{I}$$

$$\rightarrow \quad (\nabla_{\boldsymbol{r}} \boldsymbol{v})^s_{jk} = \frac{1}{2} (\partial_j v_k + \partial_k v_j) - \delta_{jk} \frac{1}{3} \partial_m v_m$$

$$(\nabla_{\boldsymbol{r}} \boldsymbol{v})^a = \frac{1}{2}\left[\nabla_{\boldsymbol{r}} \boldsymbol{v} - (\nabla_{\boldsymbol{r}} \boldsymbol{v})^T \right]$$

$$\rightarrow \quad (\nabla_{\boldsymbol{r}} \boldsymbol{v})^a_{jk} = \frac{1}{2} (\partial_j v_k - \partial_k v_j) \tag{10.27a}$$

Note that there are two orthogonal components of $\nabla_{\boldsymbol{r}} \boldsymbol{v}$ that are symmetric:

$$\frac{1}{3} (\nabla_{\boldsymbol{r}} \cdot \boldsymbol{v})\, \mathbb{I} \qquad \& \qquad (\nabla_{\boldsymbol{r}} \boldsymbol{v})^s \tag{10.27b}$$

For $\boldsymbol{v} = \boldsymbol{\Omega} \times \boldsymbol{r}$,

$$\nabla_{\boldsymbol{r}} \cdot \boldsymbol{v} = \nabla_{\boldsymbol{r}} \cdot (\boldsymbol{\Omega} \times \boldsymbol{r}) = \partial_j (\epsilon_{jmn} \Omega_m x_n) = \epsilon_{jmn} \Omega_m \delta_{jn} = \epsilon_{jmj} \Omega_m = 0$$

which means both components in (10.27b) can be included in (1c).

Since physical quantities must be independent of the system of coordinates used, they are characterized by their tensor properties. [Reminder: tensors are defined in terms of coordinate transformations of their components.] We must therefore split (1c) into two mutually orthogonal parts indicated by (10.27b) :

$$\Pi = -\zeta \,\mathrm{Tr}(\nabla_{\boldsymbol{r}} \boldsymbol{v}) = -\zeta \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v} \qquad (10.31)$$

and

$$\boldsymbol{\Pi}^s = -2\,\eta\,(\nabla_{\boldsymbol{r}} \boldsymbol{v})^s = -\eta\left[\,\nabla_{\boldsymbol{r}} \boldsymbol{v} + (\nabla_{\boldsymbol{r}} \boldsymbol{v})^T\,\right] \qquad (10.30)$$

where ζ is the coefficient of bulk viscosity and η the coefficient of shear viscosity.

In terms of the decomposition

$$\boldsymbol{\Pi} = \frac{1}{3}\mathrm{Tr}(\boldsymbol{\Pi})\,\mathbb{I} + \boldsymbol{\Pi}^s + \boldsymbol{\Pi}^a \qquad (10.28)$$

(10.30-1) implies

$$\Pi = \frac{1}{3}\mathrm{Tr}(\boldsymbol{\Pi}) \qquad \& \qquad \boldsymbol{\Pi}^a = 0$$

Consider now the entropy source (10.26). In particular

$$\begin{aligned}\left(\boldsymbol{\Pi}^T \cdot \nabla_{\boldsymbol{r}}\right) \cdot \boldsymbol{v} = \Pi_{jk}\,\partial_j v_k &= \boldsymbol{\Pi} : (\nabla_{\boldsymbol{r}} \boldsymbol{v}) \\ &= \Pi\,(\nabla_{\boldsymbol{r}} \cdot \boldsymbol{v}) + \boldsymbol{\Pi}^s : (\nabla_{\boldsymbol{r}} \boldsymbol{v})^s \qquad \text{[(7) used.]} \\ &= -\zeta\,(\nabla_{\boldsymbol{r}} \cdot \boldsymbol{v})^2 - 2\,\eta\,(\nabla_{\boldsymbol{r}} \boldsymbol{v})^s : (\nabla_{\boldsymbol{r}} \boldsymbol{v})^s \qquad \text{[(10.30-1) used.]}\end{aligned}$$

If we define

$$\boldsymbol{J}_s^D = -\frac{K}{T}\nabla_{\boldsymbol{r}} T \qquad (10.29)$$

then (10.26) can be written as

$$\begin{aligned}\sigma_s &= \frac{K}{T}(\nabla_{\boldsymbol{r}} T)\cdot(\nabla_{\boldsymbol{r}} T) + \frac{1}{T}\zeta\,(\nabla_{\boldsymbol{r}} \cdot \boldsymbol{v})^2 + 2\,\frac{\eta}{T}(\nabla_{\boldsymbol{r}} \boldsymbol{v})^s : (\nabla_{\boldsymbol{r}} \boldsymbol{v})^s \\ &= \frac{K}{T}\,|\nabla_{\boldsymbol{r}} T|^2 + \frac{1}{T}\zeta\,(\nabla_{\boldsymbol{r}} \cdot \boldsymbol{v})^2 + 2\,\frac{\eta}{T}\,|(\nabla_{\boldsymbol{r}} \boldsymbol{v})^s|^2 \qquad (10.32)\end{aligned}$$

where

$$|\nabla_{\boldsymbol{r}} T|^2 \equiv (\nabla_{\boldsymbol{r}} T)\cdot(\nabla_{\boldsymbol{r}} T) \qquad |(\nabla_{\boldsymbol{r}} \boldsymbol{v})^s|^2 \equiv (\nabla_{\boldsymbol{r}} \boldsymbol{v})^s : (\nabla_{\boldsymbol{r}} \boldsymbol{v})^s$$

On the other hand, (10.28) gives

$$\begin{aligned}\nabla_{\boldsymbol{r}} \cdot \boldsymbol{\Pi} &= \nabla_{\boldsymbol{r}} \cdot (\Pi\,\mathbb{I}) + \nabla_{\boldsymbol{r}} \cdot \boldsymbol{\Pi}^s \\ &= -\zeta\,\nabla_{\boldsymbol{r}}\left(\nabla_{\boldsymbol{r}} \cdot \boldsymbol{v}\right) - 2\,\eta\,\nabla_{\boldsymbol{r}} \cdot (\nabla_{\boldsymbol{r}} \boldsymbol{v})^s \qquad (10.32a)\end{aligned}$$

Using

$$\begin{aligned}\nabla_{\boldsymbol{r}} \cdot (\nabla_{\boldsymbol{r}} \boldsymbol{v})^s = \partial_j (\nabla_{\boldsymbol{r}} \boldsymbol{v})^s_{jk} &= \partial_j\left[\frac{1}{2}(\partial_j v_k + \partial_k v_j) - \delta_{jk}\frac{1}{3}\partial_m v_m\right] \qquad \text{[(10.27a) used.]} \\ &= \frac{1}{2}\left[\,\nabla_{\boldsymbol{r}}^2 \boldsymbol{v} + \nabla_{\boldsymbol{r}}(\nabla_{\boldsymbol{r}} \cdot \boldsymbol{v})\,\right] - \frac{1}{3}\nabla_{\boldsymbol{r}}(\nabla_{\boldsymbol{r}} \cdot \boldsymbol{v})\,\Big] \\ &= \frac{1}{2}\nabla_{\boldsymbol{r}}^2 \boldsymbol{v} + \frac{1}{6}\nabla_{\boldsymbol{r}}(\nabla_{\boldsymbol{r}} \cdot \boldsymbol{v})\end{aligned}$$

(10.32a) becomes

$$\nabla_{\boldsymbol{r}} \cdot \boldsymbol{\Pi} = -\left(\zeta + \frac{1}{3}\eta\right)\nabla_{\boldsymbol{r}}\left(\nabla_{\boldsymbol{r}} \cdot \boldsymbol{v}\right) - \eta\,\nabla_{\boldsymbol{r}}^2 \boldsymbol{v}$$

The momentum balance equation (10.8) thus becomes

$$\frac{\partial\,\rho\,\boldsymbol{v}}{\partial\,t} + \nabla_{\boldsymbol{r}} \cdot (\rho\,\boldsymbol{v}\,\boldsymbol{v}) = \rho\,\boldsymbol{F} - \nabla_{\boldsymbol{r}} P + \left(\zeta + \frac{1}{3}\eta\right)\nabla_{\boldsymbol{r}}\left(\nabla_{\boldsymbol{r}} \cdot \boldsymbol{v}\right) + \eta\,\nabla_{\boldsymbol{r}}^2 \boldsymbol{v} \qquad (10.34)$$

where we have also used

$$\nabla_{\boldsymbol{r}} \cdot (P\,\mathbb{I}) = \nabla_{\boldsymbol{r}} P$$

Replacing σ_s with (10.32), the entropy balance equation (10.24) becomes

$$\frac{\partial(\rho s)}{\partial t} + \nabla_{\boldsymbol{r}} \cdot \left(\rho s \boldsymbol{v} + \boldsymbol{J}_s^D\right) = \frac{K}{T} |\nabla_{\boldsymbol{r}} T|^2 + \frac{1}{T} \zeta (\nabla_{\boldsymbol{r}} \cdot \boldsymbol{v})^2 + 2 \frac{\eta}{T} |(\nabla_{\boldsymbol{r}} \boldsymbol{v})^s|^2 \qquad (10.35)$$

For completeness, we also write down the mass balance equation (10.3)

$$\frac{\partial \rho}{\partial t} + \nabla_{\boldsymbol{r}} \cdot (\rho \boldsymbol{v}) = 0 \qquad (10.33)$$

Solutions to the 5 scalar equations in (10.33-5) completely determine the state of the fluid at all times. They, or (10.34) alone, are called the Navier-Stokes equations.

Useful Mathematics -- Tensors

Caution: Implicit summation is suspended in this section.

Any tensor

$$\mathbb{T} = T_{xx} \hat{\boldsymbol{x}} \hat{\boldsymbol{x}} + T_{xy} \hat{\boldsymbol{x}} \hat{\boldsymbol{y}} + \ldots + T_{zz} \hat{\boldsymbol{z}} \hat{\boldsymbol{z}} \qquad (1)$$

can be decomposed into 3 orthogonal components

$$\mathbb{T} = \frac{1}{3} (\mathrm{Tr}\, \mathbb{T})\, \mathbb{I} + \mathbb{T}^s + \mathbb{T}^a \qquad (2)$$

where

$$\mathbb{I} = \hat{\boldsymbol{x}} \hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} \hat{\boldsymbol{y}} + \hat{\boldsymbol{z}} \hat{\boldsymbol{z}} = \text{unit tensor}$$

and the trace

$$\mathrm{Tr}\, \mathbb{T} = \sum_j T_{jj} = \text{sum of all diagonal elements of } \mathbb{T} \qquad (2a)$$

$\mathbb{T}^s$ is a traceless symmetric tensor defined as

$$\mathbb{T}^s = \frac{1}{2} \left(\mathbb{T} + \mathbb{T}^T\right) - \frac{1}{3} (\mathrm{Tr}\, \mathbb{T})\, \mathbb{I} \qquad (3)$$

while $\mathbb{T}^a$ is an anti-symmetric tensor defined as

$$\mathbb{T}^a = \frac{1}{2} \left(\mathbb{T} - \mathbb{T}^T\right) \qquad (5)$$

According to (3), the components of $\mathbb{T}^s$ are given by

$$\begin{aligned} T_{jk}^s &= \frac{1}{2} (T_{jk} + T_{kj}) - \delta_{jk} \frac{1}{3} \sum_{m=1}^{3} T_{mm} \\ &= \begin{cases} \dfrac{2}{3} T_{jj} - \dfrac{1}{3} \displaystyle\sum_{m \neq j} T_{mm} & \text{if } j = k \\ \dfrac{1}{2} (T_{jk} + T_{kj}) & \text{if } j \neq k \end{cases} && (4) \\ &= T_{kj}^s \qquad \text{(symmetric)} && (4a) \end{aligned}$$

Hence,

$$\begin{aligned} \mathrm{Tr}\, \mathbb{T}^s &= \sum_{j=1}^{3} \left(\frac{2}{3} T_{jj} - \frac{1}{3} \sum_{m \neq j} T_{mm} \right) = \frac{2}{3} \mathrm{Tr}\, \mathbb{T} - \frac{1}{3} \sum_{j=1}^{3} (\mathrm{Tr}\, \mathbb{T} - T_{jj}) \\ &= \frac{2}{3} \mathrm{Tr}\, \mathbb{T} - \frac{1}{3} (3\, \mathrm{Tr}\, \mathbb{T} - \mathrm{Tr}\, \mathbb{T}) = 0 \qquad \text{(traceless)} && (4b) \end{aligned}$$

Similarly, (5) gives

$$T^a_{jk} = \frac{1}{2}(T_{jk} - T_{kj}) = -T^a_{kj} \qquad \text{(anti-symmetric)} \qquad (4c)$$

$$\rightarrow \quad T^a_{jj} = 0 \qquad (6)$$

Given another tensor $\mathbb{V}$, we can form the scalar product

$$\mathbb{T} : \mathbb{V} \equiv \sum_{jk} T_{jk} V_{jk} = \mathbb{V} : \mathbb{T} \qquad (7a)$$

As in the case of scalar product between vectors, two tensors $\mathbb{T}$ & $\mathbb{V}$ are orthogonal if

$$\mathbb{T} : \mathbb{V} = 0$$

For example,

$$\mathbb{T}^s : \mathbb{I} = \sum_{jk} T^s_{jk} \delta_{jk} = \sum_j T^s_{jj} = \text{Tr}\, \mathbb{T}^s = 0 \qquad \text{[(4b) used.]}$$

$$\mathbb{T}^a : \mathbb{I} = \sum_{jk} T^a_{jk} \delta_{jk} = \sum_j T^a_{jj} = 0 \qquad \text{[(6) used.]}$$

$$\mathbb{T}^s : \mathbb{T}^a = \sum_{jk} T^s_{jk} T^a_{jk} = \sum_{jk} T^s_{kj} T^a_{kj} \qquad (j \leftrightarrow k)$$

$$= \sum_{jk} T^s_{kj} \left(-T^a_{kj}\right) \qquad \text{[(4a,c) used.]}$$

$$= 0 \qquad (6a)$$

Hence, the 3 terms in (2) are mutually orthogonal, as stated.

(6a) can be easily generalized to

$$\mathbb{T}^s : \mathbb{V}^a = 0 \qquad (6b)$$

Using (2), we have

$$\mathbb{T} : \mathbb{V} = \left[\frac{1}{3}(\text{Tr}\, \mathbb{T})\, \mathbb{I} + \mathbb{T}^s + \mathbb{T}^a\right] : \left[\frac{1}{3}(\text{Tr}\, \mathbb{V})\, \mathbb{I} + \mathbb{V}^s + \mathbb{V}^a\right]$$

$$= \frac{1}{9}(\text{Tr}\, \mathbb{T})(\text{Tr}\, \mathbb{V})\, \mathbb{I} : \mathbb{I} + \mathbb{T}^s : \mathbb{V}^s + \mathbb{T}^a : \mathbb{V}^a \qquad \text{[Orthogonality used.]}$$

$$= \frac{1}{3}(\text{Tr}\, \mathbb{T})(\text{Tr}\, \mathbb{V}) + \mathbb{T}^s : \mathbb{V}^s + \mathbb{T}^a : \mathbb{V}^a \qquad (7)$$

where

$$\mathbb{I} : \mathbb{I} = \sum_{jk} \delta_{jk} \delta_{jk} = \sum_j \delta_{jj} = 3$$

was used.

Thus, one can replace the 9 Cartesian components $T_{\alpha\beta}$ of $\mathbb{T}$ with the three symmetrized components $\frac{1}{3}(\text{Tr}\, \mathbb{T})$, $\mathbb{T}^s$ & $\mathbb{T}^a$. Note that the number of scalar components remains 9 since there are 5 & 3 independent components in $\mathbb{T}^s$ & $\mathbb{T}^a$, respectively.