10.C.1. Linearization of the Hydrodynamic Equations

For processes close to the equilibrium state, the hydrodynamic equations (10.33-5) can be linearized.

Using subscript 0 to denote the equilibrium value, we can write

$$X(\boldsymbol{r}, t) = X_0 + \Delta X(\boldsymbol{r}, t) \tag{10.36a}$$

for any thermodynamic variable X such as ρ, T, s, P, etc. For processes near equilibrium, we expect

$$|\,\Delta X\,| << |\,X_0\,|$$

so that only terms $O[\,\Delta X\,]$ need be considered.

By definition, thermodynamic variables of the equilibrium state are homogenous and time-independent, i.e.,

$$X_0 = \text{constant}$$

For fluids, equilibrium means $\boldsymbol{F} = 0$ & $\boldsymbol{v}_0 = \boldsymbol{0}$ so that $|\,\boldsymbol{F}\,| \ll 1$ & $|\,\boldsymbol{v}\,| \ll 1$ for processes near equilibrium.

In the following, we shall linearize the hydrodynamic equations (10.33-5) around the equilibrium point. However, the results should be easily adapted to the linearization about a steady state with $\boldsymbol{F} \neq 0$ & $\boldsymbol{v}_0 \neq \boldsymbol{0}$.

Using

$$\frac{\partial \rho}{\partial t} = \frac{\partial \Delta\rho}{\partial t}$$

$$\nabla_{\boldsymbol{r}} \cdot (\rho\, \boldsymbol{v}) \approx \nabla_{\boldsymbol{r}} \cdot (\rho_0\, \boldsymbol{v}) = \rho_0 \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v}$$

(10.33) is linearized as

$$\frac{\partial \Delta\rho}{\partial t} + \rho_0 \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v} = 0$$

(10.36)

Similarly, the linear version of the Navier-Stokes (momentum) equation (10.34) is

$$\rho_0 \frac{\partial \boldsymbol{v}}{\partial t} = \rho_0 \boldsymbol{F} - \nabla_{\boldsymbol{r}} \Delta P + \left(\zeta + \frac{1}{3}\eta\right) \nabla_{\boldsymbol{r}} \left(\nabla_{\boldsymbol{r}} \cdot \boldsymbol{v}\right) + \eta \nabla_{\boldsymbol{r}}^2 \boldsymbol{v} \tag{10.37}$$

Finally, using

$$\rho\, s \approx \rho_0\, s_0 + \rho_0\, \Delta s + s_0\, \Delta\rho$$

$$\rho\, s\, \boldsymbol{v} \approx \rho_0\, s_0\, \boldsymbol{v}$$

$$\boldsymbol{J}_s^D = -\frac{K}{T} \nabla_{\boldsymbol{r}} T \approx -\frac{K}{T_0} \nabla_{\boldsymbol{r}} \Delta T \qquad \text{[(10.29) used.]}$$

$$|\,\nabla_{\boldsymbol{r}} T\,|^2 = (\nabla_{\boldsymbol{r}} \Delta T) \cdot (\nabla_{\boldsymbol{r}} \Delta T) \approx 0$$

$$(\nabla_{\boldsymbol{r}} \cdot \boldsymbol{v})^2 \approx 0 \qquad |\,(\nabla_{\boldsymbol{r}} \boldsymbol{v})^s\,|^2 \approx 0$$

the entropy balance equation (10.35) linearizes as

$$\rho_0 \frac{\partial \Delta s}{\partial t} + s_0 \frac{\partial \Delta\rho}{\partial t} + \rho_0\, s_0 \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v} - \frac{K}{T_0} \nabla_{\boldsymbol{r}}^2 \Delta T = 0$$

which can be further simplified using (10.36) to

$$\rho_0 \frac{\partial \Delta s}{\partial t} = \frac{K}{T_0} \nabla_{\boldsymbol{r}}^2 \Delta T$$

(10.38)

As already stated, there are only 5 independent variables among the 5 scalar equations in (10.36-8). Choosing them to be $\boldsymbol{v}$, $\Delta\rho$ & ΔT, the other 2 variables can be written as

$$\Delta s = \left(\frac{\partial s}{\partial \rho}\right)_T^0 \Delta\rho + \left(\frac{\partial s}{\partial T}\right)_\rho^0 \Delta T \qquad (10.39)$$

$$\Delta P = \left(\frac{\partial P}{\partial \rho}\right)_T^0 \Delta\rho + \left(\frac{\partial P}{\partial T}\right)_\rho^0 \Delta T \qquad (10.40)$$

(10.36-8) thus become

$$\frac{\partial \Delta\rho}{\partial t} + \rho_0 \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v} = 0 \qquad \text{[Same as 10.36).]}$$

(10.41)

$$\rho_0 \frac{\partial \boldsymbol{v}}{\partial t} = \rho_0 \boldsymbol{F} - \left(\frac{\partial P}{\partial \rho}\right)_T^0 \nabla_{\boldsymbol{r}} \Delta\rho - \left(\frac{\partial P}{\partial T}\right)_\rho^0 \nabla_{\boldsymbol{r}} \Delta T + \left(\zeta + \frac{1}{3}\eta\right) \nabla_{\boldsymbol{r}} \left(\nabla_{\boldsymbol{r}} \cdot \boldsymbol{v}\right) + \eta \nabla_{\boldsymbol{r}}^2 \boldsymbol{v} \qquad (10.42)$$

$$\rho_0 \left(\frac{\partial s}{\partial \rho}\right)_T^0 \frac{\partial \Delta\rho}{\partial t} + \rho_0 \left(\frac{\partial s}{\partial T}\right)_\rho^0 \frac{\partial \Delta T}{\partial t} = \frac{K}{T_0} \nabla_{\boldsymbol{r}}^2 \Delta T \qquad (10.43)$$

Ideal Fluid

For an ideal fluid, there is no dissipation so that

$$\boldsymbol{J}_s^D = 0, \quad \boldsymbol{\Pi} = 0 \quad \rightarrow \quad K = 0, \quad \eta = \zeta = 0 \qquad \text{[See (10.29-31).]}$$

(10.42-3) thus simplify to

$$\rho_0 \frac{\partial \boldsymbol{v}}{\partial t} = \rho_0 \boldsymbol{F} - \left(\frac{\partial P}{\partial \rho}\right)_T^0 \nabla_{\boldsymbol{r}} \Delta\rho - \left(\frac{\partial P}{\partial T}\right)_\rho^0 \nabla_{\boldsymbol{r}} \Delta T \qquad (10.45)$$

$$\left(\frac{\partial s}{\partial \rho}\right)_T^0 \frac{\partial \Delta\rho}{\partial t} + \left(\frac{\partial s}{\partial T}\right)_\rho^0 \frac{\partial \Delta T}{\partial t} = 0$$

(10.44a)

Using

$$\frac{\left(\frac{\partial s}{\partial \rho}\right)_T^0}{\left(\frac{\partial s}{\partial T}\right)_\rho^0} = -\left(\frac{\partial T}{\partial \rho}\right)_s^0 \qquad \text{[See (2.6).]}$$

(10.44a) becomes

$$\frac{\partial \Delta T}{\partial t} = \left(\frac{\partial T}{\partial \rho}\right)_s^0 \frac{\partial \Delta\rho}{\partial t}$$

(10.44)

Assuming $\boldsymbol{F}$ is time-independent, $\frac{\partial}{\partial t}$(10.45) gives

$$\rho_0 \frac{\partial^2 \boldsymbol{v}}{\partial t^2} = -\left(\frac{\partial P}{\partial \rho}\right)_T^0 \nabla_{\boldsymbol{r}} \frac{\partial \Delta\rho}{\partial t} - \left(\frac{\partial P}{\partial T}\right)_\rho^0 \nabla_{\boldsymbol{r}} \frac{\partial \Delta T}{\partial t}$$

$$= -\left[\left(\frac{\partial P}{\partial \rho}\right)_T^0 + \left(\frac{\partial P}{\partial T}\right)_\rho^0 \left(\frac{\partial T}{\partial \rho}\right)_s^0\right] \nabla_{\boldsymbol{r}} \frac{\partial \Delta\rho}{\partial t} \qquad \text{[(10.44) used.]}$$

$$= -\left(\frac{\partial P}{\partial \rho}\right)_s^0 \nabla_{\boldsymbol{r}} \frac{\partial \Delta\rho}{\partial t} \qquad \text{[(2.8) used.]}$$

$$= \left(\frac{\partial P}{\partial \rho}\right)_s^0 \rho_0 \nabla_{\boldsymbol{r}} (\nabla_{\boldsymbol{r}} \cdot \boldsymbol{v}) \qquad \text{[(10.41) used.]}$$

$$\rightarrow \qquad \frac{\partial^2 \mathbf{v}}{\partial t^2} - \left(\frac{\partial P}{\partial \rho}\right)_s^0 \nabla_{\mathbf{r}} (\nabla_{\mathbf{r}} \cdot \mathbf{v}) = 0$$

$$\frac{\partial^2 \mathbf{v}}{\partial t^2} - \left(\frac{\partial P}{\partial \rho}\right)_s^0 \left[\nabla_{\mathbf{r}}^2 \mathbf{v} + \nabla_{\mathbf{r}} \times (\nabla_{\mathbf{r}} \times \mathbf{v}) \right] = 0$$

(10.46)

Consider now a plane wave solution to (10.46)

$$\mathbf{v}(\mathbf{r}, t) = \mathbf{v}_{\mathbf{k}} \, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

$$\rightarrow \qquad -\omega^2 \mathbf{v}_{\mathbf{k}} + \left(\frac{\partial P}{\partial \rho}\right)_s^0 \left[k^2 \mathbf{v}_{\mathbf{k}} + \mathbf{k} \times (\mathbf{k} \times \mathbf{v}_{\mathbf{k}}) \right] = 0$$

(10.46a)

Setting

$$\mathbf{v}_{\mathbf{k}} = v_{\mathbf{k}}^{\parallel} \hat{\mathbf{k}} + \mathbf{v}_{\mathbf{k}}^{\perp} \qquad \text{with} \qquad \hat{\mathbf{k}} \cdot \mathbf{v}_{\mathbf{k}}^{\perp} = 0$$

(10.50)

we have

$$v_k^{\parallel} = \hat{\mathbf{k}} \cdot \mathbf{v}_{\mathbf{k}}$$

and

$$\begin{aligned} \mathbf{k} \times (\mathbf{k} \times \mathbf{v}_{\mathbf{k}}) = \mathbf{k} \times \left(\mathbf{k} \times \mathbf{v}_{\mathbf{k}}^{\perp}\right) &= \epsilon_{ijm} \, \epsilon_{mnl} \, k_j \, k_n \, v_{\mathbf{k}l}^{\perp} \\ &= (\delta_{in} \, \delta_{jl} - \delta_{il} \, \delta_{jn}) \, k_j \, k_n \, v_{\mathbf{k}l}^{\perp} = k_j \, k_i \, v_{\mathbf{k}j}^{\perp} - k_n \, k_n \, v_{\mathbf{k}i}^{\perp} \\ &= -k^2 \, \mathbf{v}_{\mathbf{k}}^{\perp} \qquad \text{[(10.50) used.]} \end{aligned}$$

(10.46a) thus becomes

$$-\omega^2 \mathbf{v}_{\mathbf{k}}^{\perp} + \left[\omega^2 - \left(\frac{\partial P}{\partial \rho}\right)_s^0 k^2\right] v_{\mathbf{k}}^{\parallel} \hat{\mathbf{k}} = 0 \qquad (10.46b)$$

which gives

$$\mathbf{v}_{\mathbf{k}}^{\perp} = 0 \qquad \& \qquad \omega^2 = \left(\frac{\partial P}{\partial \rho}\right)_s^0 k^2$$

The solution to (10.46),

$$\mathbf{v}(\mathbf{r}, t) = v_{\mathbf{k}}^{\parallel} \hat{\mathbf{k}} \, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

is a longitudinal sound wave with speed (and phase velocity)

$$c = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_s^0} = \frac{1}{\sqrt{\rho \, \kappa_s}}$$

(10.46c)

where the adiabatic compressibility is defined as

$$\kappa_s \equiv -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_s = -\rho \left(\frac{\partial}{\partial P} \frac{1}{\rho}\right)_s = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial P}\right)_s$$

(10.44-6) also suggests that ideal fluids are characterized by adiabatic processes.

Dissipative Fluid

Using the spatial Fourier transforms

$$\Delta X(\mathbf{r}, t) = \int \frac{d^3 k}{(2\pi)^3} \, e^{i\mathbf{k}\cdot\mathbf{r}} \, X_{\mathbf{k}}(t) \qquad (10.47a)$$

$$X_{\boldsymbol{k}}(t) = \int d^3 r\, e^{-i \boldsymbol{k}\cdot\boldsymbol{r}}\, \Delta X(\boldsymbol{r}, t)$$

the linearized equations (10.41-3) can be re-written, for $\boldsymbol{F} = -\nabla_{\boldsymbol{r}}\, \phi$, as

$$\frac{d \rho_{\boldsymbol{k}}}{d t} + i\, \rho_0\, \boldsymbol{k}\cdot\boldsymbol{v}_{\boldsymbol{k}} = 0$$

(10.47)

$$\rho_0 \frac{d \boldsymbol{v}_{\boldsymbol{k}}}{d t} = -i\, \rho_0\, \boldsymbol{k}\, \phi_{\boldsymbol{k}} - i \left(\frac{\partial P}{\partial \rho}\right)_T^0 \boldsymbol{k}\, \rho_{\boldsymbol{k}} - i \left(\frac{\partial P}{\partial T}\right)_\rho^0 \boldsymbol{k}\, T_{\boldsymbol{k}} - \left(\zeta + \frac{1}{3}\eta\right) \boldsymbol{k} \left(\boldsymbol{k}\cdot\boldsymbol{v}_{\boldsymbol{k}}\right) - \eta\, k^2\, \boldsymbol{v}_{\boldsymbol{k}} \qquad (10.48)$$

$$\rho_0 \left(\frac{\partial s}{\partial \rho}\right)_T^0 \frac{d \rho_{\boldsymbol{k}}}{d t} + \rho_0 \left(\frac{\partial s}{\partial T}\right)_\rho^0 \frac{d T_{\boldsymbol{k}}}{d t} = -\frac{K}{T_0} k^2\, T_{\boldsymbol{k}} \qquad (10.49)$$

Caution: From this point onwards, our equations are related to Reichl's by $\boldsymbol{k} \to -\boldsymbol{k}$ since $X_{\boldsymbol{k}}(t) = X_{-\boldsymbol{k}}^{\text{Reichl}}(t)$.

Separating $\boldsymbol{v}_{\boldsymbol{k}}$ into longitudinal & perpendicular components [see (10.50)], we have

$$\frac{d \rho_{\boldsymbol{k}}}{d t} + i\, \rho_0\, k\, v_{\boldsymbol{k}}^{||} = 0$$

(10.51)

$$\rho_0 \frac{d \boldsymbol{v}_{\boldsymbol{k}}^{//}}{d t} = -i \rho_0\, k\, \phi_{\boldsymbol{k}} - i \left(\frac{\partial P}{\partial \rho}\right)_T^0 k\, \rho_{\boldsymbol{k}} - i \left(\frac{\partial P}{\partial T}\right)_\rho^0 k\, T_{\boldsymbol{k}} - \left(\zeta + \frac{4}{3}\eta\right) k^2\, \boldsymbol{v}_{\boldsymbol{k}}^{//} \qquad (10.52)$$

$$\rho_0 \frac{d \boldsymbol{v}_{\boldsymbol{k}}^{\perp}}{d t} = -\eta\, k^2\, \boldsymbol{v}_{\boldsymbol{k}}^{\perp}$$

(10.54)

$$\rho_0 \left(\frac{\partial s}{\partial \rho}\right)_T^0 \frac{d \rho_{\boldsymbol{k}}}{d t} + \rho_0 \left(\frac{\partial s}{\partial T}\right)_\rho^0 \frac{d T_{\boldsymbol{k}}}{d t} = -\frac{K}{T_0} k^2\, T_{\boldsymbol{k}} \qquad \text{[Same as (10.49).]} \qquad (10.53)$$

Owing to decays caused by dissipation, the time-dependence of the variables must be handled by Laplace transforms

$$X_{\boldsymbol{k}}(t) = \frac{1}{2\pi i} \int_{-i\infty+\delta}^{i\infty+\delta} d z\, e^{z t}\, \tilde{X}_{\boldsymbol{k}}(z)$$

$$\tilde{X}_{\boldsymbol{k}}(z) = \int_0^\infty d t\, e^{-z t}\, X_{\boldsymbol{k}}(t)$$

(10.55)

where $\delta \geq 0$ is chosen such that all possible singularities of $e^{z t}\, \tilde{X}_{\boldsymbol{k}}(z)$ lie to the left of the (vertical) line of integration. Consider now the contour obtained by joining the ends of this line with a semicircle lying on the left hand side. Since $e^{z t}\, \tilde{X}_{\boldsymbol{k}}(z) = 0$ on the semicircle, where $\text{Re}\, z = -\infty$, we have

$$X_{\boldsymbol{k}}(t) = \sum \text{Residue}\left[e^{z t}\, \tilde{X}_{\boldsymbol{k}}(z)\right]$$

Taking the time derivative of (10.55) gives

$$\frac{d}{d t} \int_0^\infty d t\, e^{-z t}\, X_{\boldsymbol{k}}(t) = e^{-z t}\, X_{\boldsymbol{k}}(t)\Big|_0^\infty = -X_{\boldsymbol{k}}(0)$$

$$= \int_0^\infty d t\, e^{-z t} \left[-z\, X_{\boldsymbol{k}}(t) + \frac{d X_k}{d t}\right]$$

$$\rightarrow \quad \int_0^\infty d t\, e^{-z t} \frac{d X_k}{d t} = -X_{\boldsymbol{k}}(0) + z\, \tilde{X}_{\boldsymbol{k}}(z)$$

(10.56)

Taking the Laplace transform of (10.51-4) then gives

$$z\, \tilde{\rho}_{\boldsymbol{k}}(z) + i\, \rho_0\, k\, \tilde{v}_{\boldsymbol{k}}^{||}(z) = \rho_{\boldsymbol{k}}(0) \qquad (10.57)$$

$$\rho_0\, z\, \tilde{\boldsymbol{v}}_{\boldsymbol{k}}^{//}(z) + i\,\rho_0\, k\, \phi_{\boldsymbol{k}} + i\left(\frac{\partial P}{\partial \rho}\right)_T^0 k\,\tilde{\rho}_{\boldsymbol{k}}(z) + i\left(\frac{\partial P}{\partial T}\right)_\rho^0 k\,\tilde{T}_{\boldsymbol{k}}(z) + \left(\zeta + \frac{4}{3}\eta\right) k^2\,\tilde{\boldsymbol{v}}_{\boldsymbol{k}}^{//}(z) = \rho_0\,\boldsymbol{v}_{\boldsymbol{k}}^{//}(0) \qquad (10.58)$$

$$\rho_0\, z\, \tilde{\boldsymbol{v}}_{\boldsymbol{k}}^{\perp}(z) + \eta\, k^2\, \tilde{\boldsymbol{v}}_{\boldsymbol{k}}^{\perp}(z) = \rho_0\,\boldsymbol{v}_{\boldsymbol{k}}^{\perp}(0)$$

(10.60)

$$\rho_0\left(\frac{\partial s}{\partial \rho}\right)_T^0 z\,\tilde{\rho}_{\boldsymbol{k}}(z) + \rho_0\left(\frac{\partial s}{\partial T}\right)_\rho^0 z\,\tilde{T}_{\boldsymbol{k}}(z) + \frac{K}{T_0} k^2\,\tilde{T}_{\boldsymbol{k}}(z) = \rho_0\left(\frac{\partial s}{\partial \rho}\right)_T^0 \rho_{\boldsymbol{k}}(0) + \rho_0\left(\frac{\partial s}{\partial T}\right)_\rho^0 T_{\boldsymbol{k}}(0) \qquad (10.59)$$

The partials between thermodynamic variables can be written in terms of material parameters.

To begin, with

$$\tilde{c}_P = T\left(\frac{\partial s}{\partial T}\right)_P \qquad \tilde{c}_\rho = T\left(\frac{\partial s}{\partial T}\right)_\rho$$

(2.152) gives

$$\gamma \equiv \frac{\tilde{c}_P}{\tilde{c}_\rho} = \frac{\left(\frac{\partial \rho}{\partial P}\right)_T}{\left(\frac{\partial \rho}{\partial P}\right)_s} = c^2\left(\frac{\partial \rho}{\partial P}\right)_T$$

[(10.46c) used.]

$$\rightarrow \qquad \left(\frac{\partial P}{\partial \rho}\right)_T = \frac{c^2}{\gamma}$$

(10.61)

Next, (2.144) gives

$$\tilde{c}_P - \tilde{c}_\rho = T\left(\frac{\partial s}{\partial \rho}\right)_T \left(\frac{\partial \rho}{\partial T}\right)_P$$

(10.61a)

Using

$$\left(\frac{\partial s}{\partial \rho}\right)_T = -\frac{1}{\rho^2}\left(\frac{\partial P}{\partial T}\right)_\rho$$

[Maxwell relation from diagram

$$\begin{matrix} P/\rho^2 & ---- & T \\ | & \nearrow & | \\ s & \underset{u}{----} & \rho \end{matrix} .$$

Arrow from ρ to P/ρ^2 not shown due to lack of symbol.]

$$\alpha_P = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_P = \text{thermal expansivity}.$$

(10.61a) becomes

$$\tilde{c}_P - \tilde{c}_\rho = \frac{T\,\alpha_P}{\rho}\left(\frac{\partial P}{\partial T}\right)_\rho \qquad (10.61b)$$

Finally, (2.6) gives

$$\left(\frac{\partial P}{\partial T}\right)_\rho = -\frac{\left(\frac{\partial \rho}{\partial T}\right)_P}{\left(\frac{\partial \rho}{\partial P}\right)_T} = \frac{\rho\,\alpha_P\,c^2}{\gamma}$$

(10.61c)

so that

$$\left(\frac{\partial s}{\partial \rho}\right)_T = -\frac{\alpha_P\,c^2}{\rho\,\gamma}$$

(10.61d)

We also define

$$\nu_l = \frac{1}{\rho}\left(\zeta + \frac{4}{3}\eta\right) = \text{longitudinal kinetic viscosity}.$$

$$\nu_t = \frac{\eta}{\rho} = \text{transverse kinetic viscosity}.$$

$$\chi = \frac{K}{\rho\, \tilde{c}_P} = \text{thermal diffusivity}. \qquad (10.62)$$

With the understanding that these material parameters are taken at the equilibrium state, we can write (10.57-60) as

$$z\, \tilde{\rho}_{\boldsymbol{k}}(z) + i\, \rho_0\, k\, \tilde{v}^{||}_{\boldsymbol{k}}(z) = \rho_{\boldsymbol{k}}(0) \qquad \text{[Same as (10.57).]} \qquad (10.63)$$

$$\left(z + \nu_l\, k^2\right) \tilde{\boldsymbol{v}}^{//}_{\boldsymbol{k}}(z) + i\, k\, \phi_{\boldsymbol{k}} + i\, k\, \frac{c^2}{\gamma\, \rho_0}\, \tilde{\rho}_{\boldsymbol{k}}(z) + i\, k\, \frac{c^2\, \alpha_P}{\gamma}\, \tilde{T}_{\boldsymbol{k}}(z) = \boldsymbol{v}^{//}_{\boldsymbol{k}}(0) \qquad (10.64)$$

$$\left(z + \nu_t\, k^2\right) \tilde{\boldsymbol{v}}^{\perp}_{\boldsymbol{k}}(z) = \boldsymbol{v}^{\perp}_{\boldsymbol{k}}(0)$$

(10.66)

$$-\frac{\alpha_P\, c^2}{\rho_0\, \gamma}\, z\, \tilde{\rho}_{\boldsymbol{k}}(z) + \frac{\tilde{c}_\rho}{T_0}\left(z + \gamma\, \chi\, k^2\right) \tilde{T}_{\boldsymbol{k}}(z) = -\frac{\alpha_P\, c^2}{\rho_0\, \gamma}\, \rho_{\boldsymbol{k}}(0) + \frac{\tilde{c}_\rho}{T_0}\, T_{\boldsymbol{k}}(0) \qquad (10.65)$$