10.C.3. Longitudinal Hydrodynamic Modes

The equations (10.63-5) for the longitudinal modes can be written in matrix form as

$$\mathbb{M}\begin{pmatrix}\tilde{\rho}_{\boldsymbol{k}}(z)\\ \tilde{\boldsymbol{v}}^{//}_{\boldsymbol{k}}(z)\\ \tilde{T}_{\boldsymbol{k}}(z)\end{pmatrix} \equiv \begin{pmatrix} z & i\rho_0 k & 0\\ ik\dfrac{c^2}{\gamma\rho_0} & z+\nu_l k^2 & ik\dfrac{c^2\alpha_P}{\gamma}\\ -\dfrac{\alpha_P c^2}{\rho_0\gamma}z & 0 & \dfrac{\tilde{c}_\rho}{T_0}\left(z+\gamma\chi k^2\right)\end{pmatrix}\begin{pmatrix}\tilde{\rho}_{\boldsymbol{k}}(z)\\ \tilde{\boldsymbol{v}}^{//}_{\boldsymbol{k}}(z)\\ \tilde{T}_{\boldsymbol{k}}(z)\end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ -\dfrac{\alpha_P c^2}{\rho_0\gamma} & 0 & \dfrac{\tilde{c}_\rho}{T_0}\end{pmatrix}\begin{pmatrix}\rho_{\boldsymbol{k}}(0)\\ \boldsymbol{v}^{//}_{\boldsymbol{k}}(0)\\ T_{\boldsymbol{k}}(0)\end{pmatrix} + \begin{pmatrix}0\\ -ik\phi_{\boldsymbol{k}}\\ 0\end{pmatrix} \tag{10.69}$$

$$= \begin{pmatrix}\rho_{\boldsymbol{k}}(0)\\ \boldsymbol{v}^{//}_{\boldsymbol{k}}(0) - ik\phi_{\boldsymbol{k}}\\ -\dfrac{\alpha_P c^2}{\rho_0\gamma}\rho_{\boldsymbol{k}}(0) + \dfrac{\tilde{c}_\rho}{T_0}T_{\boldsymbol{k}}(0)\end{pmatrix}$$

(10.69a)

Hence,

$$\begin{pmatrix}\tilde{\rho}_{\boldsymbol{k}}(z)\\ \tilde{\boldsymbol{v}}^{//}_{\boldsymbol{k}}(z)\\ \tilde{T}_{\boldsymbol{k}}(z)\end{pmatrix} = \mathbb{M}^{-1}\begin{pmatrix}\rho_{\boldsymbol{k}}(0)\\ \boldsymbol{v}^{//}_{\boldsymbol{k}}(0) - ik\phi_{\boldsymbol{k}}\\ -\dfrac{\alpha_P c^2}{\rho_0\gamma}\rho_{\boldsymbol{k}}(0) + \dfrac{\tilde{c}_\rho}{T_0}T_{\boldsymbol{k}}(0)\end{pmatrix}$$

(10.70)

The inverse of $\mathbb{M}$ can be written as

$$\mathbb{M}^{-1} = \frac{1}{\mathcal{D}}\mathbb{C}^T$$

(10.70a)

where $\mathcal{D} = \det\mathbb{M}$ and $\mathbb{C}$ is the cofactor matrix of $\mathbb{M}$ with elements

$$C_{ij} = (-)^{i+j}\det\mathbb{A}_{ij}$$

where $\mathbb{A}_{ij}$ is the matrix obtained by deleting the i^{th} row & j^{th} column of $\mathbb{M}$.

Using the Laplace expansion on the 1st row of $\mathbb{M}$, we have

$$\mathcal{D} = z\left(z+\nu_l k^2\right)\frac{\tilde{c}_\rho}{T_0}\left(z+\gamma\chi k^2\right) - i\rho_0 k\left[ik\frac{c^2}{\gamma\rho_0}\frac{\tilde{c}_\rho}{T_0}\left(z+\gamma\chi k^2\right) + \frac{ik}{\rho_0}\left(\frac{c^2\alpha_P}{\gamma}\right)^2 z\right] \tag{10.71a}$$

Combining (10.61b-c), we get

$$\tilde{c}_P - \tilde{c}_\rho = \frac{T_0\alpha_P^2 c^2}{\gamma}$$

$$\rightarrow \qquad c^2\alpha_P^2 = \frac{\tilde{c}_\rho}{T_0}\gamma(\gamma-1)$$

(10.71b)

(10.71a) then simplifies to

$$\mathcal{D} = z\left(z + \nu_l k^2\right)\frac{\tilde{c}_\rho}{T_0}\left(z + \gamma\chi k^2\right) + k^2 c^2 \frac{\tilde{c}_\rho}{T_0}\left(\chi k^2 + z\right)$$
$$= \frac{\tilde{c}_\rho}{T_0}\left[z^3 + (\nu_l + \gamma\chi) k^2 z^2 + \left(\nu_l \gamma \chi k^4 + c^2 k^2\right) z + c^2 \chi k^4\right] \qquad (10.71)$$

Also,

$$C_{11} = \begin{vmatrix} z + \nu_l k^2 & i k \frac{c^2 \alpha_P}{\gamma} \\ 0 & \frac{\tilde{c}_\rho}{T_0}\left(z + \gamma\chi k^2\right) \end{vmatrix} = \left(z + \nu_l k^2\right)\frac{\tilde{c}_\rho}{T_0}\left(z + \gamma\chi k^2\right)$$

$$C_{12} = -\begin{vmatrix} i k \frac{c^2}{\gamma \rho_0} & i k \frac{c^2 \alpha_P}{\gamma} \\ -\frac{\alpha_P c^2}{\rho_0 \gamma} z & \frac{\tilde{c}_\rho}{T_0}\left(z + \gamma\chi k^2\right) \end{vmatrix} = -i k \frac{c^2}{\gamma \rho_0}\frac{\tilde{c}_\rho}{T_0}\left(z + \gamma\chi k^2\right) - \frac{i k}{\rho_0}\left(\frac{c^2 \alpha_P}{\gamma}\right)^2 z$$
$$= -i k c^2 \frac{\tilde{c}_\rho}{\rho_0 T_0}\left(\chi k^2 + z\right)$$

$$C_{13} = \begin{vmatrix} i k \frac{c^2}{\gamma \rho_0} & z + \nu_l k^2 \\ -\frac{\alpha_P c^2}{\rho_0 \gamma} z & 0 \end{vmatrix} = \frac{\alpha_P c^2}{\rho_0 \gamma} z\left(z + \nu_l k^2\right)$$

$$C_{21} = -\begin{vmatrix} i \rho_0 k & 0 \\ 0 & \frac{\tilde{c}_\rho}{T_0}\left(z + \gamma\chi k^2\right) \end{vmatrix} = -i \rho_0 k \frac{\tilde{c}_\rho}{T_0}\left(z + \gamma\chi k^2\right)$$

$$C_{22} = \begin{vmatrix} z & 0 \\ -\frac{\alpha_P c^2}{\rho_0 \gamma} z & \frac{\tilde{c}_\rho}{T_0}\left(z + \gamma\chi k^2\right) \end{vmatrix} = \frac{\tilde{c}_\rho}{T_0}\left(z + \gamma\chi k^2\right) z$$

$$C_{23} = -\begin{vmatrix} z & i \rho_0 k \\ -\frac{\alpha_P c^2}{\rho_0 \gamma} z & 0 \end{vmatrix} = -i k \frac{\alpha_P c^2}{\gamma} z$$

$$C_{31} = \begin{vmatrix} i \rho_0 k & 0 \\ z + \nu_l k^2 & i k \frac{c^2 \alpha_P}{\gamma} \end{vmatrix} = -\rho_0 k^2 \frac{c^2 \alpha_P}{\gamma}$$

$$C_{32} = -\begin{vmatrix} z & 0 \\ i k \frac{c^2}{\gamma \rho_0} & i k \frac{c^2 \alpha_P}{\gamma} \end{vmatrix} = -i k \frac{c^2 \alpha_P}{\gamma} z$$

$$C_{33}=\begin{vmatrix} z & i\rho_0 k \\ ik\dfrac{c^2}{\gamma\rho_0} & z+\nu_l k^2 \end{vmatrix} = z\left(z+\nu_l k^2\right)+k^2\frac{c^2}{\gamma} = z^2+\nu_l k^2 z+k^2\frac{c^2}{\gamma}$$

Hence

$$\mathbb{C}^T=\begin{pmatrix} \left(z+\nu_l k^2\right)\dfrac{\tilde{c}_\rho}{T_0}\left(z+\gamma\chi k^2\right) & -i\rho_0 k\dfrac{\tilde{c}_\rho}{T_0}\left(z+\gamma\chi k^2\right) & -\rho_0 k^2\dfrac{c^2\alpha_P}{\gamma} \\ -ikc^2\dfrac{\tilde{c}_\rho}{\rho_0 T_0}\left(\chi k^2+z\right) & \dfrac{\tilde{c}_\rho}{T_0}\left(z+\gamma\chi k^2\right)z & -ik\dfrac{c^2\alpha_P}{\gamma}z \\ \dfrac{\alpha_P c^2}{\rho_0\gamma}z\left(z+\nu_l k^2\right) & -ik\dfrac{\alpha_P c^2}{\gamma}z & z^2+\nu_l k^2 z+k^2\dfrac{c^2}{\gamma} \end{pmatrix} \tag{10.72}$$

Consider now the homogeneous equation

$$\mathbb{M}\begin{pmatrix} \tilde{\rho}_{\boldsymbol{k}}(z) \\ \tilde{\boldsymbol{v}}_{\boldsymbol{k}}^{//}(z) \\ \tilde{T}_{\boldsymbol{k}}(z) \end{pmatrix}=0 \tag{10.73a}$$

which corresponds to the case of no external force and the system is initially in equilibrium so that

$$\rho_{\boldsymbol{k}}(0)=\boldsymbol{v}_{\boldsymbol{k}}^{//}(0)=T_{\boldsymbol{k}}(0)$$

The necessary condition for a non-trivial solution [$\tilde{\rho}_{\boldsymbol{k}}(z)$, $\tilde{\boldsymbol{v}}_{\boldsymbol{k}}^{//}(z)$ & $\tilde{T}_{\boldsymbol{k}}(z)$ not all identically zero] is

$$\mathcal{D}=0$$

(10.73b)

Its solutions can be obtained using *Mathematica* [see §Code]. The results, up to $O\left(k^2\right)$, are

$$z_1=-\chi k^2 \qquad z_\pm=\pm ikc-\frac{1}{2}k^2\left[\nu_l+(\gamma-1)\chi\right]=\pm ikc-k^2\Gamma$$

(10.73a)

where

$$\Gamma=\frac{1}{2}\left[\nu_l+(\gamma-1)\chi\right]$$

Thus, we can write, for long wavelengths,

$$\mathcal{D}\approx\frac{\tilde{c}_\rho}{T_0}\left(z+\chi k^2\right)\left(z-ikc+k^2\Gamma\right)\left(z+ikc+k^2\Gamma\right)$$

(10.73)

Note that z_1 & $z_\pm$ represent the 3 normal longitudinal modes of $\mathbb{M}$ that can be excited from the equilibrium state.

In general, once $\tilde{\rho}_{\boldsymbol{k}}(z)$, $\tilde{\boldsymbol{v}}_{\boldsymbol{k}}^{//}(z)$ & $\tilde{T}_{\boldsymbol{k}}(z)$ are solved from (10.69) for a given initial conditions $\rho_{\boldsymbol{k}}(0)$, $\boldsymbol{v}_{\boldsymbol{k}}^{//}(0)$ & $T_{\boldsymbol{k}}(0)$, they can be inverted to give the evolution of the system:

$$\begin{pmatrix} \rho_{\boldsymbol{k}}(t) \\ \boldsymbol{v}_{\boldsymbol{k}}^{//}(t) \\ T_{\boldsymbol{k}}(t) \end{pmatrix}=\frac{1}{2\pi i}\int_{-i\infty+\delta}^{i\infty+\delta}dz\,e^{zt}\begin{pmatrix} \tilde{\rho}_{\boldsymbol{k}}(z) \\ \tilde{\boldsymbol{v}}_{\boldsymbol{k}}^{//}(z) \\ \tilde{T}_{\boldsymbol{k}}(z) \end{pmatrix} \tag{10.74}$$

Exercise 10.3

Compute $\rho_{\boldsymbol{k}}(t)$ assuming no external forces and initially $\rho_{\boldsymbol{k}}(0)\neq 0$, $\boldsymbol{v}_{\boldsymbol{k}}^{//}(0)=\boldsymbol{0}$ & $T_{\boldsymbol{k}}(0)=0$.
Write the amplitude of the evolution to lowest order in k.

Answer

Putting the initial conditions into (10.70) gives

$$\begin{pmatrix} \tilde{\rho}_{\boldsymbol{k}}(z) \\ \tilde{\boldsymbol{v}}^{//}_{\boldsymbol{k}}(z) \\ \tilde{T}_{\boldsymbol{k}}(z) \end{pmatrix} = \frac{1}{\mathcal{D}} \mathbb{C}^T \begin{pmatrix} \rho_{\boldsymbol{k}}(0) \\ 0 \\ -\dfrac{\alpha_P c^2}{\rho_0 \gamma} \rho_{\boldsymbol{k}}(0) \end{pmatrix}$$

$$\to \qquad \tilde{\rho}_{\boldsymbol{k}}(z) = \frac{1}{\mathcal{D}} \left(C_{11} - C_{31} \frac{\alpha_P c^2}{\rho_0 \gamma} \right) \rho_{\boldsymbol{k}}(0)$$

$$\approx \frac{(z + \nu_l k^2)(z + \gamma \chi k^2) + \dfrac{T_0}{\tilde{c}_p} k^2 \left(\dfrac{c^2 \alpha_P}{\gamma} \right)^2}{(z + \chi k^2)(z - i k c + k^2 \Gamma)(z + i k c + k^2 \Gamma)} \rho_{\boldsymbol{k}}(0) \qquad \text{[(10.71-2) used.]}$$

$$\approx \frac{(z + \nu_l k^2)(z + \gamma \chi k^2) + \dfrac{c^2 k^2}{\gamma} (\gamma - 1)}{(z + \chi k^2)(z - i k c + k^2 \Gamma)(z + i k c + k^2 \Gamma)} \rho_{\boldsymbol{k}}(0) \qquad \text{[(10.71b) used.]} \qquad (1)$$

Plugging this into

$$\rho_{\boldsymbol{k}}(t) = \frac{1}{2 \pi i} \int_{-i \infty + \delta}^{i \infty + \delta} d z \, e^{z t} \, \tilde{\rho}_{\boldsymbol{k}}(z) \qquad (2)$$

gives

$$\rho_{\boldsymbol{k}}(t) = \sum_j \text{Res}\left[e^{z_j t} \, \tilde{\rho}_{\boldsymbol{k}}(z_j) \right] \qquad z_j = \{z_1, z_+, z_-\}$$

$$= \rho_{\boldsymbol{k}}(0) \Bigg(e^{z t} \left(\left((z + \nu_l k^2)(z + \gamma \chi k^2) + \frac{c^2 k^2}{\gamma} (\gamma - 1) \right) \Big/ \left((z - i k c + k^2 \Gamma)(z + i k c + k^2 \Gamma) \right) \right) \Bigg|_{z = -\chi k^2}$$

$$+ e^{z t} \left. \frac{(z + \nu_l k^2)(z + \gamma \chi k^2) + \dfrac{c^2 k^2}{\gamma} (\gamma - 1)}{(z + \chi k^2)(z + i k c + k^2 \Gamma)} \right|_{z = i k c - k^2 \Gamma}$$

$$+ e^{z t} \left. \frac{(z + \nu_l k^2)(z + \gamma \chi k^2) + \dfrac{c^2 k^2}{\gamma} (\gamma - 1)}{(z + \chi k^2)(z - i k c + k^2 \Gamma)} \right|_{z = -i k c - k^2 \Gamma} \Bigg)$$

Keeping only the lowest order of k in each factor, we have

$$\rho_{\boldsymbol{k}}(t) \approx \rho_{\boldsymbol{k}}(0)\left(e^{-\chi k^2 t} \frac{\frac{c^2 k^2}{\gamma}(\gamma-1)}{(-i k c)(i k c)} + e^{(i k c - k^2 \Gamma) t} \frac{(i k c)(i k c) + \frac{c^2 k^2}{\gamma}(\gamma-1)}{(i k c)(2 i k c)} + e^{(-i k c - k^2 \Gamma) t} \frac{(-i k c)(-i k c) + \frac{c^2 k^2}{\gamma}(\gamma-1)}{(-i k c)(-2 i k c)} \right)$$

$$= \rho_{\boldsymbol{k}}(0)\left(e^{-\chi k^2 t} \frac{\gamma-1}{\gamma} + \frac{1}{2\gamma} e^{(i k c - k^2 \Gamma) t} + \frac{1}{2\gamma} e^{(-i k c - k^2 \Gamma) t} \right)$$

$$= \rho_{\boldsymbol{k}}(0)\left(e^{-\chi k^2 t} \frac{\gamma-1}{\gamma} + \frac{1}{\gamma} e^{-k^2 \Gamma t} \cos k c t \right) \tag{3}$$

Code

In[1]:= $\mathcal{D} = \frac{c_\rho}{T_0}\left(z^3 + (\nu_\ell + \gamma\chi)\, k^2 z^2 + \left(\nu_\ell \gamma \chi k^4 + c^2 k^2\right) z + c^2 \chi k^4 \right)$

Out[1]= $\frac{1}{T_0} c_\rho \left(z^3 + c^2 k^4 \chi + k^2 z^2 (\gamma\chi + \nu_\ell) + z\left(c^2 k^2 + k^4 \gamma \chi \nu_\ell\right)\right)$

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In[4]:= solz = Solve[𝒟 == 0, z];
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In[11]:= z1 = ((z /. solz[[1]]) + O[k]^3) // Normal // PowerExpand // Simplify
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Out[11]= $-k^2 \chi$

```
In[12]:= z2 = ((z /. solz[[2]]) + O[k]^3) // Normal // PowerExpand // Simplify
```

Out[12]= $-\frac{1}{2} k \left(-2 i c + k(-1+\gamma)\chi + k \nu_\ell\right)$

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In[13]:= z3 = ((z /. solz[[3]]) + O[k]^3) // Normal // PowerExpand // Simplify
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Out[13]= $-\frac{1}{2} k \left(2 i c + k(-1+\gamma)\chi + k \nu_\ell\right)$