10.D.2. Wiener-Khintchine Theorem

Consider the correlation matrix

$$\mathbb{C}_{\alpha\alpha}(t, t') \equiv \langle \boldsymbol{\alpha}(t)\, \boldsymbol{\alpha}^T(t') \rangle \tag{10.97a}$$

where the average is respect to a stationary distribution, e.g., an equilibrium distribution. As discussed in §§5.B & 10.D.1, $\mathbb{C}_{\alpha\alpha}$ depends only on time-differences so that

$$\mathbb{C}_{\alpha\alpha}(t, t') = \mathbb{C}_{\alpha\alpha}(t-t') = \langle \boldsymbol{\alpha}(t-t')\, \boldsymbol{\alpha}^T \rangle = \langle \boldsymbol{\alpha}\, \boldsymbol{\alpha}^T(t'-t) \rangle \tag{10.97b}$$

Setting $\tau = t - t'$, we have

$$\mathbb{C}_{\alpha\alpha}(\tau) = \langle \boldsymbol{\alpha}(\tau)\, \boldsymbol{\alpha}^T \rangle = \langle \boldsymbol{\alpha}(t+\tau)\, \boldsymbol{\alpha}^T(t) \rangle = \langle \boldsymbol{\alpha}\, \boldsymbol{\alpha}^T(-\tau) \rangle$$

$$= \langle \boldsymbol{\alpha}(-\tau)\, \boldsymbol{\alpha}^T \rangle^T = \mathbb{C}^T_{\alpha\alpha}(-\tau) \tag{10.97}$$

From (10.75), we also have from time-reversal invariance that

$$\langle \boldsymbol{\alpha}(\tau)\, \boldsymbol{\alpha}^T \rangle = \langle \boldsymbol{\alpha}\, \boldsymbol{\alpha}^T(\tau) \rangle$$

so that

$$\mathbb{C}_{\alpha\alpha}(\tau) = \mathbb{C}^T_{\alpha\alpha}(\tau) \tag{10.98}$$

$$= \mathbb{C}^T_{\alpha\alpha}(|\tau|) \qquad \text{[(10.97) used.]}$$

$$= \mathbb{C}_{\alpha\alpha}(|\tau|) \tag{10.98a}$$

i.e., $\mathbb{C}_{\alpha\alpha}$ is symmetric & depends on $|\tau|$.

According to (7.27) of §7.C,

$$\mathbb{C}_{\alpha\alpha}(0) = \langle \boldsymbol{\alpha}\, \boldsymbol{\alpha}^T \rangle = k_B\, \mathbb{g}^{-1} \tag{10.99}$$

where

$$g_{ij} = -\left(\frac{\partial^2 S}{\partial A_i\, \partial A_j} \right)_{\boldsymbol{A}=\boldsymbol{A}_0} = g_{ji} \qquad \boldsymbol{A} = \boldsymbol{A}_0 + \boldsymbol{\alpha} \tag{10.99a}$$

are related to the response functions (or generalized susceptibilities).

By definition

$$\mathbb{C}_{\alpha\alpha}(\tau) = \langle \boldsymbol{\alpha}(\tau)\, \boldsymbol{\alpha}^T \rangle = \int d^n\alpha \int d^n\alpha'\; P_2\big[\boldsymbol{\alpha}(\tau), \boldsymbol{\alpha}'\big]\, \boldsymbol{\alpha}\, \boldsymbol{\alpha}'^T$$

$$= \int d^n\alpha \int d^n\alpha'\; f(\boldsymbol{\alpha}')\, P[\boldsymbol{\alpha}' \mid \boldsymbol{\alpha}(\tau)]\, \boldsymbol{\alpha}\, \boldsymbol{\alpha}'^T$$

$$= \int d^n\alpha'\; f(\boldsymbol{\alpha}')\, \langle \boldsymbol{\alpha}(\tau) \rangle_{\boldsymbol{\alpha}'}\, \boldsymbol{\alpha}'^T \qquad \text{[(10.86) used.]}$$

$$= \int d^n\alpha'\; f(\boldsymbol{\alpha}')\, e^{-\mathbb{M}\tau} \cdot \boldsymbol{\alpha}'\, \boldsymbol{\alpha}'^T \qquad \text{[(10.88) used.]}$$

$$= e^{-\mathbb{M}\tau} \cdot \langle \boldsymbol{\alpha}\, \boldsymbol{\alpha}^T \rangle$$

$$= \mathbb{C}_{\alpha\alpha}(|\tau|) \qquad \text{[(10.98) used.]}$$

$$= e^{-\mathbb{M}|\tau|} \cdot \langle \boldsymbol{\alpha}\, \boldsymbol{\alpha}^T \rangle$$

$$= k_B\, e^{-\mathbb{M}|\tau|} \cdot \mathbb{g}^{-1} \qquad \text{[(10.99) used.]} \tag{10.100}$$

$$= k_B\, \mathbb{g}^{-1} \cdot e^{-\mathbb{M}^T|\tau|} \qquad [\, \mathbb{g}^T = \mathbb{g} \,]$$

Consider now the spectral density matrix of the fluctuations [see (5.87) of §5.E.2]

$$\mathbb{S}_{\alpha\alpha}(\omega) = \lim_{T\to\infty} \frac{1}{T}\, \boldsymbol{\alpha}(\omega; T)\, \boldsymbol{\alpha}^+(\omega; T) \tag{10.105}$$

where [see (5.86) of §5.E.2]

$$\boldsymbol{\alpha}(\omega; T) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\, \boldsymbol{\alpha}(t; T) \tag{10.103}$$

with

$$\boldsymbol{\alpha}(t\,;T) \equiv \begin{cases} \boldsymbol{\alpha}(t) & |\,t\,| < T \\ 0 & |\,t\,| > T \end{cases} \qquad (10.101)$$

so that

$$\lim_{T\to\infty} \boldsymbol{\alpha}(t\,;T) = \boldsymbol{\alpha}(t) \qquad (10.102)$$

and (10.103) can also be written as

$$\boldsymbol{\alpha}(\omega\,;T) = \int_{-T}^{T} d\,t\, e^{i\,\omega\,t}\, \boldsymbol{\alpha}(t\,;T) \qquad (10.103a)$$

Since $\boldsymbol{\alpha}(t)$ are real, (10.103a) gives

$$\boldsymbol{\alpha}^*(\omega\,;T) = \int_{-T}^{T} d\,t\, e^{-i\,\omega\,t}\, \boldsymbol{\alpha}(t\,;T) = \boldsymbol{\alpha}(-\omega\,;T) \qquad (10.104)$$

Putting (10.103) into (10.105) gives

$$\begin{aligned}
\mathbb{S}_{\alpha\alpha}(\omega) &= \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} d\,t\, e^{i\,\omega\,t}\, \boldsymbol{\alpha}(t\,;T) \int_{-\infty}^{\infty} d\,t'\, e^{-i\,\omega\,t'}\, \boldsymbol{\alpha}^T(t'\,;T) \\
&= \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} d\,t\, e^{i\,\omega\,t}\, \boldsymbol{\alpha}(t\,;T) \int_{-\infty}^{\infty} d\,\tau\, e^{-i\,\omega\,(t+\tau)}\, \boldsymbol{\alpha}^T(t+\tau;T) && [\, t' = t+\tau \,] \\
&= \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} d\,\tau\, e^{-i\,\omega\,\tau} \int_{-\infty}^{\infty} d\,t\, \boldsymbol{\alpha}(t\,;T)\, \boldsymbol{\alpha}^T(t+\tau;T) && (10.106)
\end{aligned}$$

Assuming the system to be ergodic [see §6.c], we have

$$\begin{aligned}
\left\langle \boldsymbol{\alpha}\,\boldsymbol{\alpha}^T(\tau) \right\rangle &= \lim_{T\to\infty} \frac{1}{T} \int_{-T}^{T} d\,t\, \boldsymbol{\alpha}(t)\, \boldsymbol{\alpha}^T(t+\tau) \\
&= \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} d\,t\, \boldsymbol{\alpha}(t\,;T)\, \boldsymbol{\alpha}^T(t+\tau;T) && (10.107)
\end{aligned}$$

(10.106) then becomes

$$\begin{aligned}
\mathbb{S}_{\alpha\alpha}(\omega) &= \int_{-\infty}^{\infty} d\,\tau\, e^{-i\,\omega\,\tau} \left\langle \boldsymbol{\alpha}\,\boldsymbol{\alpha}^T(\tau) \right\rangle \\
&= \int_{-\infty}^{\infty} d\,\tau\, e^{i\,\omega\,\tau} \left\langle \boldsymbol{\alpha}\,\boldsymbol{\alpha}^T(-\tau) \right\rangle = \int_{-\infty}^{\infty} d\,\tau\, e^{i\,\omega\,\tau} \left\langle \boldsymbol{\alpha}(\tau)\,\boldsymbol{\alpha}^T \right\rangle \\
&= \int_{-\infty}^{\infty} d\,\tau\, e^{i\,\omega\,\tau}\, \mathbb{C}_{\alpha\alpha}(\tau) && (10.108) \\
&= \mathbb{C}_{\alpha\alpha}(\omega)
\end{aligned}$$

i.e., the spectral density matrix is just the Fourier transform of the correlation matrix.

(10.108) is known as the Wiener-Khintchine theorem.

Taking the complex conjugate of (10.108) gives

$$\begin{aligned}
\mathbb{S}^*_{\alpha\alpha}(\omega) &= \int_{-\infty}^{\infty} d\,\tau\, e^{-i\,\omega\,\tau}\, \mathbb{C}_{\alpha\alpha}(\tau) = \mathbb{S}_{\alpha\alpha}(-\omega) && (10.110) \\
&= \int_{-\infty}^{\infty} d\,\tau\, e^{i\,\omega\,\tau}\, \mathbb{C}_{\alpha\alpha}(-\tau) && [\, \tau \to -\tau \,] \\
&= \int_{-\infty}^{\infty} d\,\tau\, e^{i\,\omega\,\tau}\, \mathbb{C}^T_{\alpha\alpha}(\tau) && [\, (10.97) \text{ used.} \,] \\
&= \mathbb{S}^T_{\alpha\alpha}(\omega) && (10.109a)
\end{aligned}$$

i.e., $\mathbb{S}_{\alpha\alpha}(\omega)$ is Hermitian

$$\mathbb{S}^+_{\alpha\alpha}(\omega) = \mathbb{S}_{\alpha\alpha}(\omega) \qquad (10.109)$$

which can be obtained directly from its definition (10.105).

Writing $\mathbb{S}_{\alpha\alpha}(\omega)$ in terms of its real & imaginary parts

$$\mathbb{S}_{\alpha\alpha}(\omega) = \mathbb{R}_{\alpha\alpha}(\omega) + i\,\mathbb{I}_{\alpha\alpha}(\omega) \qquad (10.111)$$

we have

$$\mathbb{S}_{\alpha\alpha}(-\omega) = \mathbb{R}_{\alpha\alpha}(-\omega) + i\,\mathbb{I}_{\alpha\alpha}(-\omega)$$
$$\mathbb{S}^{*}_{\alpha\alpha}(\omega) = \mathbb{R}_{\alpha\alpha}(\omega) - i\,\mathbb{I}_{\alpha\alpha}(\omega)$$
$$\mathbb{S}^{T}_{\alpha\alpha}(\omega) = \mathbb{R}^{T}_{\alpha\alpha}(\omega) + i\,\mathbb{I}^{T}_{\alpha\alpha}(\omega)$$

(10.109a) then gives

$$\mathbb{R}^{T}_{\alpha\alpha}(\omega) = \mathbb{R}_{\alpha\alpha}(\omega) \qquad \mathbb{I}^{T}_{\alpha\alpha}(\omega) = -\mathbb{I}_{\alpha\alpha}(\omega) \tag{10.111a}$$

while (10.110) gives

$$\mathbb{R}_{\alpha\alpha}(-\omega) = \mathbb{R}_{\alpha\alpha}(\omega) \qquad \mathbb{I}_{\alpha\alpha}(-\omega) = -\mathbb{I}_{\alpha\alpha}(\omega) \tag{10.111b}$$

Finally, using the time reversal invariance property (10.98a), we can write (10.108) as

$$\begin{aligned}
\mathbb{S}_{\alpha\alpha}(\omega) &= \int_0^\infty d\tau\, e^{i\omega\tau}\,\mathbb{C}_{\alpha\alpha}(\tau) + \int_{-\infty}^0 d\tau\, e^{i\omega\tau}\,\mathbb{C}_{\alpha\alpha}(-\tau) \\
&= \int_0^\infty d\tau \left(e^{i\omega\tau} + e^{-i\omega\tau}\right)\mathbb{C}_{\alpha\alpha}(\tau) \\
&= 2\int_0^\infty d\tau\, \mathbb{C}_{\alpha\alpha}(\tau)\cos\omega\tau \qquad (10.112a)
\end{aligned}$$

Since $\mathbb{C}_{\alpha\alpha}(\tau)$ is real, so is $\mathbb{S}_{\alpha\alpha}(\omega)$. Hence,

$$\mathbb{I}_{\alpha\alpha}(\omega) = 0$$

and

$$\mathbb{S}_{\alpha\alpha}(\omega) = \mathbb{R}_{\alpha\alpha}(\omega)$$

The Fourier inverse of (10.108) then gives

$$\begin{aligned}
\mathbb{C}_{\alpha\alpha}(\tau) &= \int_{-\infty}^\infty \frac{d\omega}{2\pi} e^{-i\omega\tau}\,\mathbb{S}_{\alpha\alpha}(\omega) \\
&= \int_{-\infty}^\infty \frac{d\omega}{2\pi} e^{-i\omega\tau}\,\mathbb{R}_{\alpha\alpha}(\omega) \\
&= \int_0^\infty \frac{d\omega}{2\pi} e^{-i\omega\tau}\,\mathbb{R}_{\alpha\alpha}(\omega) + \int_{-\infty}^0 \frac{d\omega}{2\pi} e^{-i\omega\tau}\,\mathbb{R}_{\alpha\alpha}(\omega) \\
&= \int_0^\infty \frac{d\omega}{2\pi} e^{-i\omega\tau}\,\mathbb{R}_{\alpha\alpha}(\omega) + \int_0^\infty \frac{d\omega}{2\pi} e^{i\omega\tau}\,\mathbb{R}_{\alpha\alpha}(-\omega) \\
&= \int_0^\infty \frac{d\omega}{2\pi} e^{-i\omega\tau}\,\mathbb{R}_{\alpha\alpha}(\omega) + \int_0^\infty \frac{d\omega}{2\pi} e^{i\omega\tau}\,\mathbb{R}_{\alpha\alpha}(\omega) \qquad \text{[(10.111b) used.]} \\
&= \int_0^\infty \frac{d\omega}{\pi}\,\mathbb{R}_{\alpha\alpha}(\omega)\cos\omega\tau \qquad (10.112) \\
&= \int_0^\infty \frac{d\omega}{\pi}\,\mathbb{S}_{\alpha\alpha}(\omega)\cos\omega\tau
\end{aligned}$$

which is just the Fourier-Cosine inverse of (10.112a).

Exercise 10.4

The dynamic density correlation function for a fluid of N particles in a box of volume V may be written as

$$C_{nn}(\boldsymbol{\rho},\tau) = \frac{1}{N}\int d\boldsymbol{r}\,\langle n(\boldsymbol{r}+\boldsymbol{\rho},\tau)\,n(\boldsymbol{r},0)\rangle \tag{1a}$$

Owing to the additional $\boldsymbol{\rho}$ dependence, the spectral density function (10.108) picks up an extra Fourier transform so that

$$S_{nn}(\boldsymbol{k},\Omega) = \int d\boldsymbol{\rho}\int_{-\infty}^\infty d\tau\, e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}+i\Omega\tau}\,C_{nn}(\boldsymbol{\rho},\tau) \tag{6}$$

which is also known as the dynamic structure factor.

Compute $C_{nn}(\boldsymbol{\rho}, \tau)$ & $S_{nn}(\boldsymbol{k}, \Omega)$ for a fluid in equilibrium. Assume that fluctuations in the average velocity is independent of both temperature and density fluctuations.

Answer (a) $C_{nn}(\boldsymbol{\rho}, \tau)$

Let

$$n(\boldsymbol{r}, t) = n_0 + \Delta n(\boldsymbol{r}, t) \qquad n_0 = \langle n(\boldsymbol{r}, t) \rangle = \frac{N}{V} \tag{1c}$$

$$\rightarrow \qquad \langle \Delta n(\boldsymbol{r}, t) \rangle = 0 \tag{1d}$$

(1a) then becomes

$$\begin{aligned} C_{nn}(\boldsymbol{\rho}, \tau) &= \frac{1}{N} \int d\boldsymbol{r} \left\langle \left[n_0 + \Delta n(\boldsymbol{r}+\boldsymbol{\rho}, \tau) \right] \left[n_0 + \Delta n(\boldsymbol{r}, 0) \right] \right\rangle \\ &= \frac{1}{N} \int d\boldsymbol{r} \left[n_0^2 + \langle \Delta n(\boldsymbol{r}+\boldsymbol{\rho}, \tau) \rangle n_0 + \langle \Delta n(\boldsymbol{r}, 0) \rangle n_0 + \langle \Delta n(\boldsymbol{r}+\boldsymbol{\rho}, \tau) \Delta n(\boldsymbol{r}, 0) \rangle \right] \\ &= n_0 + \frac{1}{N} \int d\boldsymbol{r} \langle \Delta n(\boldsymbol{r}+\boldsymbol{\rho}, \tau) \Delta n(\boldsymbol{r}, 0) \rangle \end{aligned} \tag{1}$$

Imposing periodic boundary conditions on the fluctuations gives the Fourier series

$$\Delta n(\boldsymbol{r}, t) = \frac{1}{V} \sum_{\boldsymbol{k}} e^{i \boldsymbol{k} \cdot \boldsymbol{r}} n_{\boldsymbol{k}}(t) \tag{2}$$

with inverse

$$n_{\boldsymbol{k}}(t) = \int d\boldsymbol{r}\, e^{-i \boldsymbol{k} \cdot \boldsymbol{r}} \Delta n(\boldsymbol{r}, t) \tag{2a}$$

Similarly,

$$C_{nn}(\boldsymbol{\rho}, \tau) = \frac{V}{(2\pi)^3} \int d\boldsymbol{k}\; e^{i \boldsymbol{k} \cdot \boldsymbol{\rho}} C_{nn}(\boldsymbol{k}, \tau) \tag{3a}$$

$$\begin{aligned} C_{nn}(\boldsymbol{k}, \tau) &= \int d\boldsymbol{\rho}\, e^{-i \boldsymbol{k} \cdot \boldsymbol{\rho}} C_{nn}(\boldsymbol{\rho}, \tau) \\ &= \int d\boldsymbol{\rho}\, e^{-i \boldsymbol{k} \cdot \boldsymbol{\rho}} \left[n_0 + \frac{1}{N} \int d\boldsymbol{r} \langle \Delta n(\boldsymbol{r}+\boldsymbol{\rho}, \tau) \Delta n(\boldsymbol{r}, 0) \rangle \right] \qquad [\text{ (1) used. }] \end{aligned}$$

Using

$$\int d\boldsymbol{\rho}\, e^{-i \boldsymbol{k} \cdot \boldsymbol{\rho}} = V \delta_{\boldsymbol{k}\boldsymbol{0}} = \frac{N}{n_0} \delta_{\boldsymbol{k}\boldsymbol{0}} \tag{4}$$

we have

$$\begin{aligned} C_{nn}(\boldsymbol{k}, \tau) &= N \delta_{\boldsymbol{k}\boldsymbol{0}} + \frac{1}{N} \int d\boldsymbol{\rho}\, e^{-i \boldsymbol{k} \cdot \boldsymbol{\rho}} \int d\boldsymbol{r} \langle \Delta n(\boldsymbol{r}+\boldsymbol{\rho}, \tau) \Delta n(\boldsymbol{r}, 0) \rangle \\ &= N \delta_{\boldsymbol{k}\boldsymbol{0}} + \frac{1}{N} \int d\boldsymbol{R}\, e^{-i \boldsymbol{k} \cdot \boldsymbol{R}} \int d\boldsymbol{r}\, e^{i \boldsymbol{k} \cdot \boldsymbol{r}} \langle \Delta n(\boldsymbol{R}, \tau) \Delta n(\boldsymbol{r}, 0) \rangle \qquad [\boldsymbol{R} = \boldsymbol{r} + \boldsymbol{\rho}] \\ &= N \delta_{\boldsymbol{k}\boldsymbol{0}} + \frac{1}{N} \langle n_{\boldsymbol{k}}(\tau) n_{-\boldsymbol{k}}(0) \rangle \qquad [\text{ (2a) used. }] \end{aligned} \tag{3}$$

Now, the state of the fluid is specified by $\rho = m n$, $\boldsymbol{v}$ & T and the equilibrium state by $\Delta\rho = \Delta\boldsymbol{v} = \Delta T = 0$. Since this exercise assumes the fluctuations of these variables are independent of each other, $\rho_{\boldsymbol{k}}(\tau)$ is independent of the initial values of $\boldsymbol{v}_{\boldsymbol{k}}$ & $T_{\boldsymbol{k}}$. By setting $\boldsymbol{v}_{\boldsymbol{k}}(0) = T_{\boldsymbol{k}}(0) = 0$, we can make use of the results in Exercise 10.3. Hence, to lowest order in $\boldsymbol{k}$ [see §10.C.3 for notations],

$$n_{\boldsymbol{k}}(t) = \frac{1}{m}\rho_{\boldsymbol{k}}(t) \approx n_{\boldsymbol{k}}(0)\left(e^{-\chi k^2 |t|}\,\frac{\gamma-1}{\gamma} + \frac{1}{\gamma} e^{-k^2 \Gamma |t|} \cos k c t\right) \tag{4a}$$

where we have replaced t with $|t|$ in the exponentials since $n_{\boldsymbol{k}}(t)$ must be finite for $t \to \pm\infty$.

(3) then becomes,

$$C_{n n}(\boldsymbol{k}, t) \approx N \delta_{\boldsymbol{k}\boldsymbol{0}} + \frac{1}{N}\langle n_{\boldsymbol{k}}(0)\, n_{-\boldsymbol{k}}(0)\rangle\left(e^{-\chi k^2 |t|}\,\frac{\gamma-1}{\gamma} + \frac{1}{\gamma} e^{-k^2 \Gamma |t|} \cos k c t\right) \tag{5}$$

Answer (b) $S_{n n}(\boldsymbol{k}, \Omega)$

(6) can be written as

$$S_{n n}(\boldsymbol{k}, \Omega) = \int_{-\infty}^{\infty} d\tau\, e^{i \Omega \tau} C_{n n}(\boldsymbol{k}, \tau)$$

Plugging in (5) gives

$$S_{n n}(\boldsymbol{k}, \Omega) = \int_{-\infty}^{\infty} d t\, e^{i \Omega t}\left[N \delta_{\boldsymbol{k}\boldsymbol{0}} + \frac{1}{N}\langle n_{\boldsymbol{k}}(0)\, n_{-\boldsymbol{k}}(0)\rangle\left(e^{-\chi k^2 |t|}\,\frac{\gamma-1}{\gamma} + \frac{1}{\gamma} e^{-k^2 \Gamma |t|} \cos k c t\right)\right]$$

Using

$$\int_{-\infty}^{\infty} d t\, e^{i \Omega t} = 2 \pi \delta(\Omega)$$

$$\begin{aligned}
\int_{-\infty}^{\infty} d t\, e^{i \Omega t} e^{-\chi k^2 |t|} &= \int_{0}^{\infty} d t\, e^{(i \Omega - \chi k^2) t} + \int_{-\infty}^{0} d t\, e^{(i \Omega + \chi k^2) t} \\
&= -\frac{1}{i \Omega - \chi k^2} + \frac{1}{i \Omega + \chi k^2} \\
&= \frac{2 \chi k^2}{\Omega^2 + \chi^2 k^4}
\end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^{\infty} d t\, e^{-k^2 \Gamma |t|} \cos k c t &= \int_{0}^{\infty} d t\, e^{(i \Omega - \Gamma k^2) t} \cos k c t + \int_{-\infty}^{0} d t\, e^{(i \Omega + \Gamma k^2) t} \cos k c t \\
&= \int_{0}^{\infty} d t\, e^{-\Gamma k^2 t}\left(e^{i \Omega t} + e^{-i \Omega t}\right) \cos k c t \\
&= 2 \int_{0}^{\infty} d t\, e^{-\Gamma k^2 t} \cos \Omega t \cos k c t \\
&= \frac{k^2 \Gamma}{c^2 k^2 + k^4 \Gamma^2 - 2 c k \Omega + \Omega^2} + \frac{k^2 \Gamma}{c^2 k^2 + k^4 \Gamma^2 + 2 c k \Omega + \Omega^2} \qquad \text{[See §Code.]} \\
&= \frac{k^2 \Gamma}{(\Omega - c k)^2 + k^4 \Gamma^2} + \frac{k^2 \Gamma}{(\Omega + c k)^2 + k^4 \Gamma^2}
\end{aligned}$$

we have

$$\begin{aligned}
S_{n n}(\boldsymbol{k}, \Omega) = 2 \pi N \delta(\Omega) \delta_{\boldsymbol{k}\boldsymbol{0}} + \frac{1}{N}\langle n_{\boldsymbol{k}}(0)\, n_{-\boldsymbol{k}}(0)\rangle\Bigg[&\frac{\gamma-1}{\gamma}\,\frac{2 \chi k^2}{\Omega^2 + \chi^2 k^4} \\
&+ \frac{1}{\gamma}\left(\frac{k^2 \Gamma}{(\Omega - c k)^2 + k^4 \Gamma^2} + \frac{k^2 \Gamma}{(\Omega + c k)^2 + k^4 \Gamma^2}\right)\Bigg]
\end{aligned} \tag{7}$$

The 1st term represents the dominant static The 3 terms in the square brackets are Lorentzian of the form

$$f(\Omega) = \frac{2 \Delta}{(\Omega - \Omega_0) + \Delta^2} \tag{8}$$

which denotes a distribution with a peak at Ω_0 with a half-width at half-maximum (HWHM) of Δ.

$S_{nn}(\boldsymbol{k}, \Omega)$ thus contains 3 peaks at $\Omega = 0$ & $\pm c k$. The peak at $\Omega = 0$, with half-width χk^2, is called the Rayleigh peak. Those at $\Omega = \pm c k$, with half-width Γk^2, are called Brillouin peaks.

Code

In[3]:=
```
par = {a → Γ k^2, b → Ω, d → k c};
Assuming[a > 0 && b > 0 && d > 0, ∫_0^∞ e^(-a t) Cos[b t] Cos[d t] dt] /. par
```

Out[4]=
$$\frac{k^2 \Gamma \left(c^2 k^2 + k^4 \Gamma^2 + \Omega^2\right)}{c^4 k^4 + 2 c^2 k^2 \left(k^2 \Gamma - \Omega\right) \left(k^2 \Gamma + \Omega\right) + \left(k^4 \Gamma^2 + \Omega^2\right)^2}$$

In[6]:=
```
% // Factor
```

Out[6]=
$$\frac{k^2 \Gamma \left(c^2 k^2 + k^4 \Gamma^2 + \Omega^2\right)}{\left(c^2 k^2 + k^4 \Gamma^2 - 2 c k \Omega + \Omega^2\right) \left(c^2 k^2 + k^4 \Gamma^2 + 2 c k \Omega + \Omega^2\right)}$$

In[7]:=
```
% // Apart
```

Out[7]=
$$\frac{k^2 \Gamma}{2 \left(c^2 k^2 + k^4 \Gamma^2 - 2 c k \Omega + \Omega^2\right)} + \frac{k^2 \Gamma}{2 \left(c^2 k^2 + k^4 \Gamma^2 + 2 c k \Omega + \Omega^2\right)}$$