10.E.1. The Response Matrix

Read Reichl's introduction to §10.E.

Let $\boldsymbol{F} = (F_1, \ldots, F_n)$ be the external forces that couple to the state variables

$$\boldsymbol{A} = (A_1, \ldots, A_n) = \boldsymbol{A}^0 + \boldsymbol{\alpha}_{\boldsymbol{F}}(t)$$

where $\boldsymbol{A}^0 = \left(A_1^0, \ldots, A_n^0\right)$ and $\boldsymbol{\alpha}_{\boldsymbol{F}} = (\alpha_{F1}, \ldots, \alpha_{Fn})$ are the equilibrium values and deviations (or fluctuations), respectively. Assuming linear response, i.e., $\boldsymbol{\alpha}_{\boldsymbol{F}}$ is linear in $\boldsymbol{F}$, we have

$$\boldsymbol{\alpha}_{\boldsymbol{F}}(t) = \int_{-\infty}^{\infty} d t' \, \mathbb{K}(t - t') \cdot \boldsymbol{F}(t') \qquad (10.113a)$$

$$= -\int_{\infty}^{-\infty} d\tau \, \mathbb{K}(\tau) \cdot \boldsymbol{F}(t - \tau) \qquad [\, \tau = t - t' \,]$$

$$= \int_{-\infty}^{\infty} d\tau \, \mathbb{K}(\tau) \cdot \boldsymbol{F}(t - \tau) \qquad (10.113)$$

where $\mathbb{K}$ is called the response matrix. Since both $\boldsymbol{\alpha}_{\boldsymbol{F}}$ & $\boldsymbol{F}$ are real, so is $\mathbb{K}$.

Since the system can respond only after the forces are applied (causality), we have

$$\mathbb{K}(t - t') = 0 \qquad \forall \; t - t' < 0 \qquad (10.114)$$

$$\mathbb{K}(\tau) = 0 \qquad \forall \; \tau < 0 \qquad (10.114a)$$

and (10.113) becomes

$$\boldsymbol{\alpha}_{\boldsymbol{F}}(t) = \int_{-\infty}^{t} d t' \, \mathbb{K}(t - t') \cdot \boldsymbol{F}(t') \qquad (10.113b)$$

$$= \int_{0}^{\infty} d\tau \, \mathbb{K}(\tau) \cdot \boldsymbol{F}(t - \tau) \qquad (10.113c)$$

In terms of the Fourier transforms

$$\tilde{\boldsymbol{X}}(\omega) = \int_{-\infty}^{\infty} d t \, e^{i \omega t} \boldsymbol{X}(t) \qquad (10.115a)$$

$$\boldsymbol{X}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{\boldsymbol{X}}(\omega) \qquad (10.115b)$$

where $\boldsymbol{X} = \boldsymbol{\alpha}_{\boldsymbol{F}}$, $\boldsymbol{F}$, or $\mathbb{K}$, (10.113a) becomes

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{\boldsymbol{\alpha}}_{\boldsymbol{F}}(\omega) = \int_{-\infty}^{\infty} d t' \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} e^{-i\omega(t-t')} e^{-i\omega' t'} \, \tilde{\mathbb{K}}(\omega) \cdot \tilde{\boldsymbol{F}}(\omega')$$

Using

$$\int_{-\infty}^{\infty} d t' \, e^{i(\omega - \omega')t'} = 2\pi \, \delta(\omega - \omega')$$

we have

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{\boldsymbol{\alpha}}_{\boldsymbol{F}}(\omega) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} d\omega' \, e^{-i\omega t} \delta(\omega - \omega') \, \tilde{\mathbb{K}}(\omega) \cdot \tilde{\boldsymbol{F}}(\omega')$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \, \tilde{\mathbb{K}}(\omega) \cdot \tilde{\boldsymbol{F}}(\omega)$$

$$\rightarrow \qquad \tilde{\boldsymbol{\alpha}}_{\boldsymbol{F}}(\omega) = \tilde{\mathbb{K}}(\omega) \cdot \tilde{\boldsymbol{F}}(\omega) \qquad (10.116a)$$

$$= \boldsymbol{\chi}(\omega) \cdot \tilde{\boldsymbol{F}}(\omega) \qquad (10.116)$$

where we have replaced $\tilde{\mathbb{K}}$ with $\boldsymbol{\chi}$, the usual symbol for the dynamic susceptibility. In other words,

$$\boldsymbol{\chi}(\omega) = \tilde{\mathbb{K}}(\omega) = \int_{-\infty}^{\infty} d\tau \, e^{i\omega\tau} \mathbb{K}(\tau) \qquad (10.117)$$

$$= \int_0^\infty d\tau\, e^{i\omega\tau}\, \mathbb{K}(\tau) \qquad \text{[(10.114a) used.]} \qquad (10.117a)$$