10.E.2. Causality

We now consider the effects of causality on the response matrix $\mathbb{K}(\tau)$, or equivalently but more conveniently, its Fourier transform $\boldsymbol{\chi}(\omega)$.

To begin, consider a time-independent unit force

$$\boldsymbol{F}(t) = \hat{\boldsymbol{r}}$$

so that (10.113c) becomes

$$\boldsymbol{\alpha}_{\boldsymbol{F}}(t) = \int_0^\infty d\tau\, \mathbb{K}(\tau)\cdot\hat{\boldsymbol{r}} \qquad \forall\, t \qquad (10.118a)$$

Thus, a time-independent force $\boldsymbol{F}$ induces a time-independent deviation $\boldsymbol{\alpha}_{\boldsymbol{F}}(t)$, which is expected to be finite from physical experience. Since $\hat{\boldsymbol{r}}$ can point in any direction, we conclude that

$$\int_0^\infty d\tau\, \mathbb{K}(\tau) \quad \text{must be finite.} \qquad (10.118)$$

By definition [see (10.115b)],

$$\mathbb{K}(\tau) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega\tau} \boldsymbol{\chi}(\omega)$$
$$= \oint_C \frac{dz}{2\pi} e^{-iz\tau} \boldsymbol{\chi}(z) \qquad (10.118b)$$

where $z = \omega + i\xi$ is the complex plane with ω & ξ as its real and imaginary axes, respectively. C is a contour consisting of closing the ω axis with a semi-circle at infinity that gives no contribution to the integration. Since

$$e^{-iz\tau} = e^{-i\omega\tau} e^{\xi\tau}$$

this means C must close in the

lower-half z-plane, where $\xi \to -\infty$, for $\tau > 0$

upper-half z-plane, where $\xi \to \infty$, for $\tau < 0$

The causality condition [see (10.114)]

$$\mathbb{K}(\tau) = 0 \qquad \text{for} \qquad \tau < 0$$

can thus be achieved if $\boldsymbol{\chi}(z)$ is analytic (has no poles) in the upper-half z-plane.

Note that $\boldsymbol{\chi}(z)$ can also be considered as the extension of the Fourier transform of $\mathbb{K}(\tau)$ into the complex frequency domain:

$$\boldsymbol{\chi}(z) = \int_{-\infty}^{\infty} d\tau\, e^{iz\tau}\, \mathbb{K}(\tau)$$
$$= \int_0^\infty d\tau\, e^{iz\tau}\, \mathbb{K}(\tau) \qquad (10.119)$$

As usual, the reality of $\mathbb{K}(\tau)$ means that its Fourier transform satisfies

$$\boldsymbol{\chi}^*(\omega) = \boldsymbol{\chi}(-\omega) \qquad (10.118c)$$

$$\rightarrow \qquad \boldsymbol{\chi}'(\omega) - i\boldsymbol{\chi}''(\omega) = \boldsymbol{\chi}'(-\omega) + i\boldsymbol{\chi}''(-\omega)$$

$$\therefore \qquad \boldsymbol{\chi}'(-\omega) = \boldsymbol{\chi}'(\omega) \qquad \boldsymbol{\chi}''(-\omega) = -\boldsymbol{\chi}''(\omega) \qquad (10.118d)$$

Furthermore, (10.116) requires $\boldsymbol{\chi}(\omega)$ to be finite since both $\langle \tilde{\boldsymbol{\alpha}}(\omega) \rangle_{\boldsymbol{F}}$ & $\boldsymbol{F}(\omega)$ are.

Thus, for u real, the matrix

$$\mathbb{f}(z) = \frac{\chi(z)}{z-u} \tag{10.120}$$

is analytic in the upper z-plane except for a simple pole on the real axis at $z=u$.

Let $C_\pm$ be a contour obtained from C by replacing the infinitesimal segment through u with a semi-circle centered at u and lies $\begin{matrix}\text{above}\\\text{below}\end{matrix}$ the ω-axis. Since $C_\pm$ is a counterclockwise contour that $\begin{matrix}\text{excludes}\\\text{includes}\end{matrix}$ the pole at $z=u$, we have

$$\begin{aligned}\mathbb{I}_\pm &= \oint_{C_\pm} dz\, \mathbb{f}(z)\\ &= \mathcal{P}\int_{-\infty}^{\infty} d\omega\, \mathbb{f}(\omega) + \mathbb{J}_\pm = 2\pi i \begin{pmatrix} 0 \\ \operatorname*{Res}_{z=u} \mathbb{f}(z)\end{pmatrix}\end{aligned} \tag{10.120a}$$

where $\mathcal{P}f$ is the Cauchy principal part of the function f defined by

$$\mathcal{P}f(x) = \begin{cases} f(x) & x \neq 0 \\ \text{not defined} & x = 0\end{cases} \tag{10.123a}$$

In other word, $\mathcal{P}f(x)$ is $f(x)$ with the point $x=0$ removed. Thus,

$$\mathcal{P}\int_{-\infty}^{\infty} d\omega\, \mathbb{f}(\omega) = \lim_{r\to 0}\left(\int_{-\infty}^{u-r} d\omega + \int_{u+r}^{\infty} d\omega\right)\mathbb{f}(\omega) \tag{10.123}$$

$\mathbb{J}_\pm$ is the line integral along the $\begin{matrix}\text{upper}\\\text{lower}\end{matrix}$ semi-circle centered at u :

$$\mathbb{J}_+ = \lim_{r\to 0}\int_\pi^0 d\phi\, i r e^{i\phi}\, \mathbb{f}\left(u + r e^{i\phi}\right) \qquad \left[\; z - u = r e^{i\phi} \;\to\; dz = i r e^{i\phi} d\phi \;\right]$$

$$= -\lim_{r\to 0}\int_0^\pi d\phi\, i r e^{i\phi}\, \mathbb{f}\left(u + r e^{i\phi}\right) \tag{10.120b}$$

$$\mathbb{J}_- = \lim_{r\to 0}\int_\pi^{2\pi} d\phi\, i r e^{i\phi}\, \mathbb{f}\left(u + r e^{i\phi}\right) \tag{10.120c}$$

Putting (10.120) into (10.120b-c) gives

$$\begin{aligned}\mathbb{J}_+ &= -\lim_{r\to 0}\int_0^\pi d\phi\, i r e^{i\phi}\, \frac{\chi\left(u + r e^{i\phi}\right)}{r e^{i\phi}}\\ &= -i\int_0^\pi d\phi\, \chi(u) = -i\pi\, \chi(u)\end{aligned}$$

and

$$\mathbb{J}_- = \lim_{r\to 0}\int_\pi^{2\pi} d\phi\, i r e^{i\phi}\, \frac{\chi\left(u + r e^{i\phi}\right)}{r e^{i\phi}} = i\pi\, \chi(u)$$

Together with

$$\operatorname*{Res}_{z=u} \mathbb{f}(z) = \operatorname*{Res}_{z=u} \frac{\chi(z)}{z-u} = \chi(u)$$

(10.120a) becomes

$$\mathcal{P}\int_{-\infty}^{\infty} d\omega\, \frac{\chi(\omega)}{\omega - u} + \begin{pmatrix} -i\pi\, \chi(u) \\ i\pi\, \chi(u)\end{pmatrix} = \begin{Bmatrix} 0 \\ 2\pi i \operatorname*{Res}_{z=u} \mathbb{f}(z)\end{Bmatrix}$$

$$\to \qquad \mathcal{P}\int_{-\infty}^{\infty} d\omega\, \frac{\chi(\omega)}{\omega - u} = i\pi\, \chi(u) \tag{10.124}$$

or

$$\chi(u) = \mathcal{P}\int_{-\infty}^{\infty} \frac{d\,\omega}{i\,\pi} \frac{\chi(\omega)}{\omega - u} \tag{10.125}$$

Let $\chi'(\omega)$ & $\chi''(\omega)$ be the real & imaginary part of $\chi(\omega)$, respectively. Then

$$\chi(\omega) = \chi'(\omega) + i\,\chi''(\omega) \tag{10.126}$$

and (10.125) becomes

$$\chi'(u) + i\,\chi''(u) = \mathcal{P}\int_{-\infty}^{\infty} \frac{d\,\omega}{i\,\pi} \frac{\chi'(\omega) + i\,\chi''(\omega)}{\omega - u}$$

Equating the real & imaginary parts separately gives

$$\chi'(u)) = \mathcal{P}\int_{-\infty}^{\infty} \frac{d\,\omega}{\pi} \frac{\chi''(\omega)}{\omega - u} \tag{10.127}$$

and

$$\chi''(u) = -\mathcal{P}\int_{-\infty}^{\infty} \frac{d\,\omega}{\pi} \frac{\chi'(\omega)}{\omega - u} \tag{10.128}$$

which are known as the Kramers-Kronig relations (KKR).

Since $\chi'(\omega)$ & $\chi''(\omega)$ are related by the KKR, it is necessary to have full knowledge of only one of them.

Before going further, we need to prove the identity

$$\lim_{\eta \to 0} \frac{1}{x \pm i\,\eta} = \mathcal{P}\frac{1}{x} \mp i\,\pi\,\delta(x) \tag{10.130}$$

as follows.

$$L.H.S. = \lim_{\eta \to 0} \frac{x \mp i\,\eta}{x^2 + \eta^2}$$

$$= \begin{cases} \frac{1}{x} & x \neq 0 \\ \lim_{\eta \to 0} \frac{\mp i\,\eta}{x^2 + \eta^2} = \mp i\,\pi\,\delta(x) & x = 0 \end{cases}$$

where we have used the Lorentzian representation of the delta function

$$\delta(x) = \lim_{\eta \to 0} \frac{\eta}{\pi\left(x^2 + \eta^2\right)} \quad \text{with} \qquad \int_{-\infty}^{\infty} d\,x \frac{\eta}{\pi\left(x^2 + \eta^2\right)} = 1 \ (10.137)$$

$$R.H.S. = \begin{cases} \frac{1}{x} & x \neq 0 \\ \mp i\,\pi\,\delta\,(x) & x = 0 \end{cases}$$

$$= L.H.S.$$

QED.

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(*  Normalization of Lorentzian *)
Assuming[η > 0, ∫_{-∞}^{∞} η/(π (x² + η²)) ⅆ x]
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1
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We can re-write (10.127) as

$$\chi'(\omega) = \mathcal{P}\int_{-\infty}^{\infty} \frac{d\,\omega'}{\pi} \frac{\chi''(\omega')}{\omega' - \omega} \qquad [\,(u,\,\omega) \to (\omega,\,\omega')\,]$$

$$= \lim_{\eta\to 0}\int_{-\infty}^{\infty}\frac{d\,\omega'}{\pi}\left[\frac{1}{\omega' - \omega - i\,\eta} - i\,\pi\,\delta(\omega' - \omega)\right]\boldsymbol{\chi}''(\omega') \quad \text{[(10.130) used.]}$$

$$= \lim_{\eta\to 0}\int_{-\infty}^{\infty}\frac{d\,\omega'}{\pi}\frac{\boldsymbol{\chi}''(\omega')}{\omega' - \omega - i\,\eta} - i\,\pi\,\boldsymbol{\chi}''(\omega)$$

$$\rightarrow \qquad \boldsymbol{\chi}(\omega) = \lim_{\eta\to 0}\int_{-\infty}^{\infty}\frac{d\,\omega'}{\pi}\frac{\boldsymbol{\chi}''(\omega')}{\omega' - \omega - i\,\eta} \qquad (10.129)$$

(10.115b) then gives

$$\langle\,\boldsymbol{\alpha}(t)\,\rangle_{\boldsymbol{F}} = \int_{-\infty}^{\infty}\frac{d\,\omega}{2\,\pi}\,e^{-i\,\omega\,t}\langle\,\tilde{\boldsymbol{\alpha}}(\omega)\,\rangle_{\boldsymbol{F}} \qquad (10.115)$$

$$= \int_{-\infty}^{\infty}\frac{d\,\omega}{2\,\pi}\,e^{-i\,\omega\,t}\,\boldsymbol{\chi}(\omega)\cdot\tilde{\boldsymbol{F}}(\omega) \qquad (10.139)$$

$$= \lim_{\eta\to 0}\int_{-\infty}^{\infty}\frac{d\,\omega}{2\,\pi}\,e^{-i\,\omega\,t}\int_{-\infty}^{\infty}\frac{d\,\omega'}{\pi}\frac{\boldsymbol{\chi}''(\omega')}{\omega' - \omega - i\,\eta}\cdot\tilde{\boldsymbol{F}}(\omega) \quad \text{[(10.129) used.]}$$

$$= \lim_{\eta\to 0}\int_{-\infty}^{\infty}d\,t'\int_{-\infty}^{\infty}\frac{d\,\omega}{2\,\pi}\,e^{-i\,\omega\,t}\int_{-\infty}^{\infty}\frac{d\,\omega'}{\pi}\frac{\boldsymbol{\chi}''(\omega')}{\omega' - \omega - i\,\eta}\cdot\boldsymbol{F}(t')\,e^{i\,\omega\,t'} \qquad (10.131)$$

The integral

$$I = \lim_{\eta\to 0}\int_{-\infty}^{\infty}d\,\omega\,\frac{e^{-i\,\omega\,t}}{\omega + i\,\eta}$$

can be evaluated as a $\begin{pmatrix}\text{clockwise}\\ \text{counter-clockwise}\end{pmatrix}$ contour that closes in the $\begin{pmatrix}\text{lower}\\ \text{upper}\end{pmatrix}$ complex plane with ω as the real axis for $\begin{pmatrix}t>0\\ t<0\end{pmatrix}$. Hence,

$$I = \begin{cases}-2\,\pi\,i & t>0\\ 0 & t<0\end{cases}$$

$$\rightarrow \qquad \lim_{\eta\to 0}\int_{-\infty}^{\infty}\frac{d\,\omega}{2\,\pi}\frac{i\,e^{-i\,\omega\,t}}{\omega + i\,\eta} = \theta(t) \qquad (10.132)$$

(10.131) thus becomes

$$\boldsymbol{\alpha}_{\boldsymbol{F}}(t) = -\lim_{\eta\to 0}\int_{-\infty}^{\infty}d\,t'\int_{-\infty}^{\infty}\frac{d\,\omega'}{\pi}\int_{-\infty}^{\infty}\frac{d\,\omega}{2\,\pi}\frac{e^{-i\,\omega\,(t-t')}}{\omega - \omega' + i\,\eta}\boldsymbol{\chi}''(\omega')\cdot\boldsymbol{F}(t')$$

$$= i\int_{-\infty}^{\infty}d\,t'\,\theta(t-t')\int_{-\infty}^{\infty}\frac{d\,\omega'}{\pi}\,e^{-i\,\omega'\,(t-t')}\,\boldsymbol{\chi}''(\omega')\cdot\boldsymbol{F}(t')$$

$$= i\int_{-\infty}^{t}d\,t'\int_{-\infty}^{\infty}\frac{d\,\omega}{\pi}\,e^{-i\,\omega\,(t-t')}\,\boldsymbol{\chi}''(\omega)\cdot\boldsymbol{F}(t') \quad [\,\omega'\to\omega\,] \qquad (10.133a)$$

Setting

$$\mathbb{K}''(\tau) = \int_{-\infty}^{\infty}\frac{d\,\omega}{2\,\pi}\,e^{-i\,\omega\,\tau}\,\boldsymbol{\chi}''(\omega) \qquad (10.133b)$$

(10.133a) becomes

$$\boldsymbol{\alpha}_{\boldsymbol{F}}(t) = 2\,i\int_{-\infty}^{t}d\,t'\,\mathbb{K}''(t-t')\cdot\boldsymbol{F}(t') \qquad (10.133)$$

Note that $\mathbb{K}''(\tau) \neq \operatorname{Im}\mathbb{K}(\tau)$ partly because $\operatorname{Im}\mathbb{K}(\tau) \equiv 0$. Since $\boldsymbol{\chi}''(\omega)$ is odd in ω [see (10.118d)], we have

$$\mathbb{K}''(\tau) = \int_0^\infty \frac{d\omega}{2\pi} e^{-i\omega\tau} \boldsymbol{\chi}''(\omega) - \int_0^\infty \frac{d\omega}{2\pi} e^{i\omega\tau} \boldsymbol{\chi}''(\omega)$$

$$= \int_0^\infty \frac{d\omega}{\pi i} \boldsymbol{\chi}''(\omega) \sin\omega\tau \qquad (10.133c)$$

(10.133a) can therefore be written as

$$\boldsymbol{\alpha}_{\boldsymbol{F}}(t) = 2\int_{-\infty}^t dt' \int_0^\infty \frac{d\omega}{\pi} \sin\omega(t-t')\, \boldsymbol{\chi}''(\omega)\cdot \boldsymbol{F}(t') \qquad (10.133d)$$

Step Force

Consider the step force

$$\boldsymbol{F}(t) = \boldsymbol{F}\,\theta(-t) = \begin{cases} \boldsymbol{F} & t < 0 \\ 0 & t \geq 0 \end{cases} \qquad (10.134)$$

with Fourier transform

$$\tilde{\boldsymbol{F}}(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t} \boldsymbol{F}(t)$$

$$= \boldsymbol{F}\int_{-\infty}^{0} dt\, e^{i\omega t} = \lim_{t\to-\infty} \boldsymbol{F}\frac{1-e^{i\omega t}}{i\omega}$$

In order to evaluate

$$\lim_{t\to-\infty} e^{i\omega t}$$

we set $\omega \to \omega - i\epsilon$ with $\epsilon \to 0^+$ so that

$$\lim_{t\to-\infty} e^{i\omega t} \to \lim_{t\to-\infty} e^{i\omega t+\epsilon t} = 0$$

and

$$\tilde{\boldsymbol{F}}(\omega) = \lim_{\epsilon\to 0} \boldsymbol{F}\frac{1}{i\omega+\epsilon} = -i\,\boldsymbol{F} \lim_{\epsilon\to 0}\frac{1}{\omega - i\epsilon} \qquad (10.138a)$$

$$= -i\left[\mathcal{P}\frac{1}{\omega} + i\pi\delta(\omega)\right]\boldsymbol{F} \qquad [\text{ (10.130) used. }]$$

$$= \left[\mathcal{P}\frac{1}{i\omega} + \pi\delta(\omega)\right]\boldsymbol{F} \qquad (10.138)$$

Putting (10.134) into (10.113c) gives

$$\boldsymbol{\alpha}_{\boldsymbol{F}}(t) = \int_0^\infty d\tau\, \mathbb{K}(\tau)\cdot \boldsymbol{F}\,\theta(-t+\tau)$$

$$= \begin{cases} \int_0^\infty d\tau\, \mathbb{K}(\tau)\cdot\boldsymbol{F} & t \leq 0 \\ \int_t^\infty d\tau\, \mathbb{K}(\tau)\cdot\boldsymbol{F} & t \geq 0 \end{cases} \qquad (10.146a)$$

Using (10.117a), we have

$$\boldsymbol{\alpha}_{\boldsymbol{F}}(t) = \boldsymbol{\chi}(0)\cdot\boldsymbol{F} \qquad \text{for} \qquad t \leq 0 \qquad (10.146)$$

Also,

$$\boldsymbol{\alpha}_{\boldsymbol{F}}(\infty) = \int_\infty^\infty d\tau\, \mathbb{K}(\tau)\cdot\boldsymbol{F} = 0 \qquad (10.146b)$$

As expected from physical experiences, the deviation $\boldsymbol{\alpha}_{\boldsymbol{F}}(t)$ is a constant as long as $\boldsymbol{F}$ is kept on, but decays eventually to the equilibrium value 0 after $\boldsymbol{F}$ is turned off.

Note that (10.146) is the generalization of the familiar electrostatic relation

$$\boldsymbol{P} = \chi \cdot \boldsymbol{E}$$

between the polarization $\boldsymbol{P}$ and the electric field $\boldsymbol{E}$.

Since $\alpha_F(t)$ & $\boldsymbol{F}$ are real, so should $\chi(0)$. This is ensured as follows.

$$\mathbb{K}(\tau) \text{ is real} \quad \to \quad \chi^*(\omega) = \chi(-\omega) \quad \to \quad \chi^*(0) = \chi(0) \text{ is real} \tag{10.146c}$$

Setting $u = 0$ in (10.125) gives

$$\chi(0) = \mathcal{P} \int_{-\infty}^{\infty} \frac{d\omega}{i\pi} \frac{\chi(\omega)}{\omega} \tag{10.144}$$

Alternatively, setting $u = 0$ in (10.129) gives

$$\chi(0) = \lim_{\eta \to 0} \int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \frac{\chi''(\omega')}{\omega' - i\eta}$$

$$= \mathcal{P} \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{\chi''(\omega)}{\omega} + i\chi''(0) \qquad [\text{ (10.130) used. }]$$

$$= \mathcal{P} \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{\chi''(\omega)}{\omega} \qquad [\text{ (10.146c) used. }] \tag{10.144a}$$

$$= 2\,\mathcal{P} \int_{0}^{\infty} \frac{d\omega}{\pi} \frac{\chi''(\omega)}{\omega} \tag{10.144b}$$

which may be taken as a sum rule for $\chi''(\omega)$.

Comparing (10.144a) with (10.144), we get

$$\mathcal{P} \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{\chi'(\omega)}{\omega} = 0 \tag{10.144c}$$

which can also be derived from the fact $\chi'(-\omega) = \chi'(\omega)$ [see (10.118d)].

For $t \geq 0$, (10.146a) & (10.115b) give

$$\alpha_F(t) = \int_t^{\infty} d\tau \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega\tau} \chi(\omega) \cdot \boldsymbol{F}$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega\infty} - e^{-i\omega t}}{-i\omega} \chi(\omega) \cdot \boldsymbol{F}$$

$$= \lim_{\eta \to 0^+} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \frac{e^{-i(\omega - i\eta)t}}{\omega - i\eta} \chi(\omega) \cdot \boldsymbol{F} \tag{10.147a}$$

where, as in (10.138a), we have set $\omega \to \omega - i\eta$ to get rid of the term $e^{-i\omega\infty}$.

Using (10.130), we have, for $t \geq 0$,

$$\alpha_F(t) = \lim_{\eta \to 0^+} e^{-\eta t} \left[\mathcal{P} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \frac{e^{-i\omega t}}{\omega} \chi(\omega) \cdot \boldsymbol{F} + \frac{1}{2} \chi(0) \cdot \boldsymbol{F} \right] \tag{10.147b}$$

Note that the $e^{-\eta t}$ factor is necessary to keep $\langle \alpha(\infty) \rangle_F = 0$.

To check the result, we set $t = 0$ to get

$$\alpha_F(t) = \mathcal{P} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \frac{1}{\omega} \chi(\omega) \cdot \boldsymbol{F} + \frac{1}{2} \chi(0) \cdot \boldsymbol{F}$$

$$= \chi(0) \cdot \boldsymbol{F} \qquad [\text{ (10.144) used. }]$$

in agreement with (10.146).

Alternatively, we can use (10.133d) to get

$$\boldsymbol{\alpha}_{\boldsymbol{F}}(t) = 2\int_0^\infty d\tau \int_0^\infty \frac{d\omega}{\pi} \sin\omega\tau\, \boldsymbol{\chi}''(\omega)\cdot\boldsymbol{F}(t-\tau)$$

$$= \begin{cases} 2\int_0^\infty d\tau \int_0^\infty \dfrac{d\omega}{\pi} \sin\omega\tau\, \boldsymbol{\chi}''(\omega)\cdot\boldsymbol{F} & t\le 0 \\ 2\int_t^\infty d\tau \int_0^\infty \dfrac{d\omega}{\pi} \sin\omega\tau\, \boldsymbol{\chi}''(\omega)\cdot\boldsymbol{F} & t\ge 0 \end{cases}$$

Using

$$\int_t^\infty d\tau \sin\omega\tau = -\frac{1}{\omega}\cos\omega\tau\Big|_t^\infty = \frac{\cos\omega t}{\omega}$$

we have

$$\boldsymbol{\alpha}_{\boldsymbol{F}}(t) = \begin{cases} 2\mathcal{P}\int_0^\infty \dfrac{d\omega}{\pi} \dfrac{\boldsymbol{\chi}''(\omega)}{\omega}\cdot\boldsymbol{F} & t\le 0 \\ 2\mathcal{P}\int_0^\infty \dfrac{d\omega}{\pi} \dfrac{\cos\omega t}{\omega} \boldsymbol{\chi}''(\omega)\cdot\boldsymbol{F} & t\ge 0 \end{cases} \tag{10.147c}$$

Only the principal parts of the integrals are included since the $\omega = 0$ term gives no contribution due to the $\sin\omega\tau$ factor.

Using (10.144b), we have,

$$\boldsymbol{\alpha}_{\boldsymbol{F}}(t) = \boldsymbol{\chi}(0)\cdot\boldsymbol{F} \qquad \text{for} \qquad t\le 0$$

in agreement with (10.146).

For $t \ge 0$,

$$\boldsymbol{\alpha}_{\boldsymbol{F}}(t) = 2\mathcal{P}\int_0^\infty \frac{d\omega}{\pi} \frac{\cos\omega t}{\omega} \boldsymbol{\chi}''(\omega)\cdot\boldsymbol{F}$$

$$= \mathcal{P}\int_{-\infty}^\infty \frac{d\omega}{\pi} \cos\omega t\, \frac{\boldsymbol{\chi}''(\omega)}{\omega}\cdot\boldsymbol{F} \tag{10.147}$$