10.E.4. Power Absorption

Consider a system with extensive variable $\boldsymbol{A} = \boldsymbol{A}_0 + \boldsymbol{\alpha}$ coupled to an external force (or intensive variable) $\boldsymbol{F}$. The work done on the system by $\boldsymbol{F}$ is [see §2.D.2 or Exercise 10.5]

$$-dW = \boldsymbol{F} \cdot d\boldsymbol{\alpha} \tag{10.155}$$

where dW is the work done by the system. The power absorption of the system is therefore

$$P(t) = -\frac{dW}{dt} = \boldsymbol{F}(t) \cdot \dot{\boldsymbol{\alpha}}_F(t) \quad \text{[Dot product between vectors.]}$$

$$= \boldsymbol{F}^T(t) \cdot \dot{\boldsymbol{\alpha}}_F(t) \quad \text{[Matrix multiplication between row \& column matrices.]}$$

$$= \boldsymbol{F}^T(t) \cdot \frac{d}{dt} \int_{-\infty}^{\infty} dt' \, \mathbb{K}(t-t') \cdot \boldsymbol{F}(t') \quad \text{[(10.113) used.]} \tag{10.156}$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{\boldsymbol{F}}^T(\omega) \cdot \frac{d}{dt} \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} e^{-i\omega'(t-t')} \boldsymbol{\chi}(\omega') \cdot \int_{-\infty}^{\infty} \frac{d\omega''}{2\pi} e^{-i\omega'' t'} \tilde{\boldsymbol{F}}(\omega'')$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega''}{2\pi} (-i\omega') e^{-i(\omega+\omega')t} e^{-i(\omega''-\omega')t'} \tilde{\boldsymbol{F}}^T(\omega) \cdot \boldsymbol{\chi}(\omega') \cdot \tilde{\boldsymbol{F}}(\omega'')$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \int_{-\infty}^{\infty} d\omega'' (-i\omega') \delta(\omega''-\omega') e^{-i(\omega+\omega')t} \tilde{\boldsymbol{F}}^T(\omega) \cdot \boldsymbol{\chi}(\omega') \cdot \tilde{\boldsymbol{F}}(\omega'')$$

$$= -i \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \omega' e^{-i(\omega+\omega')t} \tilde{\boldsymbol{F}}^T(\omega) \cdot \boldsymbol{\chi}(\omega') \cdot \tilde{\boldsymbol{F}}(\omega') \tag{10.157}$$

10.E.4.1. Delta Function Force

Let

$$\boldsymbol{F}(t) = \boldsymbol{F}\,\delta(t) \quad \rightarrow \quad \tilde{\boldsymbol{F}}(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t} \boldsymbol{F}\,\delta(t) = \boldsymbol{F} \tag{10.158}$$

(10.157) becomes

$$P(t) = -i \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \omega' e^{-i(\omega+\omega')t} \boldsymbol{F}^T \cdot \boldsymbol{\chi}(\omega') \cdot \boldsymbol{F} \tag{10.159}$$

$$= -i\,\delta(t) \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \omega' e^{-i\omega' t} \boldsymbol{F}^T \cdot \boldsymbol{\chi}(\omega') \cdot \boldsymbol{F}$$

$$= -i\,\delta(t)\, \boldsymbol{F}^T \cdot \left[\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \omega\, \boldsymbol{\chi}(\omega) \right] \cdot \boldsymbol{F}$$

$$= \delta(t)\, \boldsymbol{F}^T \cdot \left[\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \omega\, \boldsymbol{\chi}''(\omega) \right] \cdot \boldsymbol{F} \quad [\, \boldsymbol{\chi}'(-\omega) = \boldsymbol{\chi}'(\omega) \,] \tag{10.159a}$$

$$= \delta(t)\, \boldsymbol{F}^T \cdot \left[\int_{0}^{\infty} \frac{d\omega}{\pi} \omega\, \boldsymbol{\chi}''(\omega) \right] \cdot \boldsymbol{F} \quad [\, \boldsymbol{\chi}''(-\omega) = -\boldsymbol{\chi}''(\omega) \,] \tag{10.159b}$$

The total energy absorbed is therefore

$$W_{abs} = \int_{-\infty}^{\infty} dt\, P(t) = \boldsymbol{F}^T \cdot \left[\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \omega\, \boldsymbol{\chi}''(\omega) \right] \cdot \boldsymbol{F} \tag{10.160}$$

$$= \boldsymbol{F}^T \cdot \left[\int_{0}^{\infty} \frac{d\omega}{\pi} \omega\, \boldsymbol{\chi}''(\omega) \right] \cdot \boldsymbol{F}$$

For a dissipative system [see Exercise 10.6],

$$\omega\, \boldsymbol{\chi}''(\omega) \geq 0 \qquad \forall\ \omega$$

$$\boldsymbol{\chi}''(\omega) \geq 0 \qquad \forall\ \omega \geq 0$$

therefore,

$$W_{abs} > 0$$

so that the system absorbs energy, part of which will turn into heat.

10.E.4.2. Oscillating Force

Let

$$\boldsymbol{F}(t) = \boldsymbol{F}\cos\omega_0 t = \frac{1}{2}\boldsymbol{F}\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right) \qquad (10.161)$$

$$\rightarrow \qquad \tilde{\boldsymbol{F}}(\omega) = \frac{1}{2}\boldsymbol{F}\int_{-\infty}^{\infty} dt\, e^{i\omega t}\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right)$$

$$= \pi\boldsymbol{F}\left[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right] \qquad (10.162)$$

(10.157) becomes

$$P(t) = -\frac{i}{4}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega'\,\omega'\, e^{-i(\omega+\omega')t}\,\boldsymbol{F}^T\cdot\boldsymbol{\chi}(\omega')\cdot\boldsymbol{F}$$

$$\times\left[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right]\left[\delta(\omega'+\omega_0) + \delta(\omega'-\omega_0)\right]$$

$$= -\frac{i}{4}\int_{-\infty}^{\infty} d\omega'\,\omega'\left(e^{-i(-\omega_0+\omega')t} + e^{-i(\omega_0+\omega')t}\right)\boldsymbol{F}^T\cdot\boldsymbol{\chi}(\omega')\cdot\boldsymbol{F}\left[\delta(\omega'+\omega_0) + \delta(\omega'-\omega_0)\right]$$

$$= -\frac{i}{4}\boldsymbol{F}^T\cdot\left[-\omega_0\left(e^{2i\omega_0 t} + 1\right)\boldsymbol{\chi}(-\omega_0) + \omega_0\left(1 + e^{-2i\omega_0 t}\right)\boldsymbol{\chi}(\omega_0)\right]\cdot\boldsymbol{F} \qquad (10.163)$$

$$= \frac{1}{2}\boldsymbol{F}^T\cdot\mathrm{Im}\left[\omega_0\left(1 + e^{-2i\omega_0 t}\right)\boldsymbol{\chi}(\omega_0)\right]\cdot\boldsymbol{F} \qquad (10.163a)$$

Averaging over one period of oscillation, we have

$$\langle P(t)\rangle = \frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0} dt\,\frac{1}{2}\boldsymbol{F}^T\cdot\mathrm{Im}\left[\omega_0\left(1 + e^{-2i\omega_0 t}\right)\boldsymbol{\chi}(\omega_0)\right]\cdot\boldsymbol{F}$$

$$= \frac{1}{2}\boldsymbol{F}^T\cdot\mathrm{Im}\left[\omega_0\,\boldsymbol{\chi}(\omega_0)\right]\cdot\boldsymbol{F}$$

$$= \frac{1}{2}\omega_0\,\boldsymbol{F}^T\cdot\boldsymbol{\chi}''(\omega_0)\cdot\boldsymbol{F} \qquad (10.164)$$

Similar to the case of the delta function force, $\langle P(t)\rangle > 0$ for a dissipative system so that it absorbs power.

Exercise 10.6

An 1-D Brownian particle of mass m is attached to a harmonic spring of force constant k, and is driven by an external force $F(t)$. The Langevin equation is

$$m\frac{d^2x(t)}{dt^2} + \gamma\frac{dx(t)}{dt} + m\,\omega_0^2\,x(t) = \xi(t) + F(t) \qquad (1a)$$

where $\omega_0 = \sqrt{\frac{k}{m}}$, γ is the friction constant, and $\xi(t)$ is a Gaussian white noise with zero mean.

Hence,

$$m\frac{d^2\langle x(t)\rangle_F}{dt^2} + \gamma\frac{d\langle x(t)\rangle_F}{dt} + m\,\omega_0^2\langle x(t)\rangle_F = F(t) \qquad (2)$$

(a) Compute and plot the linear response function $K(t)$, where

$$\langle x(t)\rangle_F = \int_{-\infty}^{\infty} dt'\,K(t-t')\,F(t') \qquad (1b)$$

(b) Compute the total energy absorbed for the case $F(t) = F_0\,\delta(t)$.

Answer (a)

By causality,

$$K(t) = 0\,\forall\; t < 0 \tag{1c}$$

Since K is independent of $F(t)$, we can set $F(t) = F_0\,\delta(t)$ so that (1c) becomes

$$\langle x(t)\rangle_F = \int_{-\infty}^{t} d\,t'\,K(t-t')\,F_0\,\delta(t') \qquad \forall\; t>0$$
$$= K(t)\,F_0 \tag{1}$$

Putting (1) into (2) gives

$$m\,\frac{d^2\,K(t)}{d\,t^2} + \gamma\,\frac{d\,K(t)}{d\,t} + m\,\omega_0^2\,K(t) = \delta(t) \tag{3}$$

which is often used as the defining equation of $K(t)$.

Taking the Fourier transform, (3) becomes

$$-m\,\omega^2\,\chi(\omega) - i\,\omega\,\gamma\,\chi(\omega) + m\,\omega_0^2\,\chi(\omega) = 1 \tag{4}$$

$$\to \qquad \chi(\omega) = \frac{1}{m\left(\omega_0^2 - \omega^2\right) - i\,\gamma\,\omega} \tag{5}$$

with

$$\chi'(\omega) = \frac{m\left(\omega_0^2-\omega^2\right)}{m^2\left(\omega_0^2-\omega^2\right)^2 + \gamma^2\,\omega^2} \tag{7}$$

$$\chi''(\omega) = \frac{\gamma\,\omega}{m^2\left(\omega_0^2-\omega^2\right)^2 + \gamma^2\,\omega^2} \tag{8}$$

The inverse transform of (5) gives

$$K(t) = \int_{-\infty}^{\infty} \frac{d\,\omega}{2\,\pi}\,\frac{e^{-i\,\omega\,t}}{m\left(\omega_0^2-\omega^2\right) - i\,\gamma\,\omega} \tag{9}$$

where the poles of the integrand are at

$$\omega_\pm = \frac{1}{2\,m}\left(-i\,\gamma \pm \sqrt{-\gamma^2 + 4\,m^2\,\omega_0^2}\right) \tag{9a}$$

Using

$$\text{Res}\left[\frac{e^{-i\,\omega\,t}}{m\left(\omega_0^2-\omega^2\right) - i\,\gamma\,\omega};\,\omega_\pm\right] = \left.\frac{e^{-i\,\omega\,t}}{-2\,m\,\omega - i\,\gamma}\right|_{\omega=\omega_\pm} = \mp\,\frac{e^{-i\,\omega_\pm\,t}}{\sqrt{-\gamma^2 + 4\,m^2\,\omega_0^2}}$$

Hence, for $t > 0$,

$$K(t) = -i\left\{\text{Res}\left[\frac{e^{-i\,\omega\,t}}{m\left(\omega_0^2-\omega^2\right) - i\,\gamma\,\omega};\,\omega_+\right] + \text{Res}\left[\frac{e^{-i\,\omega\,t}}{m\left(\omega_0^2-\omega^2\right) - i\,\gamma\,\omega};\,\omega_-\right]\right\}$$

$$= -i\,\frac{-e^{-i\,\omega_+\,t} + e^{-i\,\omega_-\,t}}{\sqrt{-\gamma^2 + 4\,m^2\,\omega_0^2}}$$

$$= \frac{2\sin\sqrt{-\frac{\gamma^2}{4\,m^2} + \omega_0^2}\;t}{\sqrt{-\gamma^2 + 4\,m^2\,\omega_0^2}}\,e^{-\gamma\,t/2\,m} \tag{10}$$

Combining with (1c), we have

$$K(t) = \theta(t)\,\frac{\sin\Omega\,t}{m\,\Omega}\,e^{-\Gamma\,t} \tag{10a}$$

where

$$\Omega = \sqrt{-\frac{\gamma^2}{4\,m^2} + \omega_0^2} = \sqrt{-\Gamma^2 + \omega_0^2} \qquad \Gamma = \frac{\gamma}{2\,m} \qquad (10b)$$

Thus,

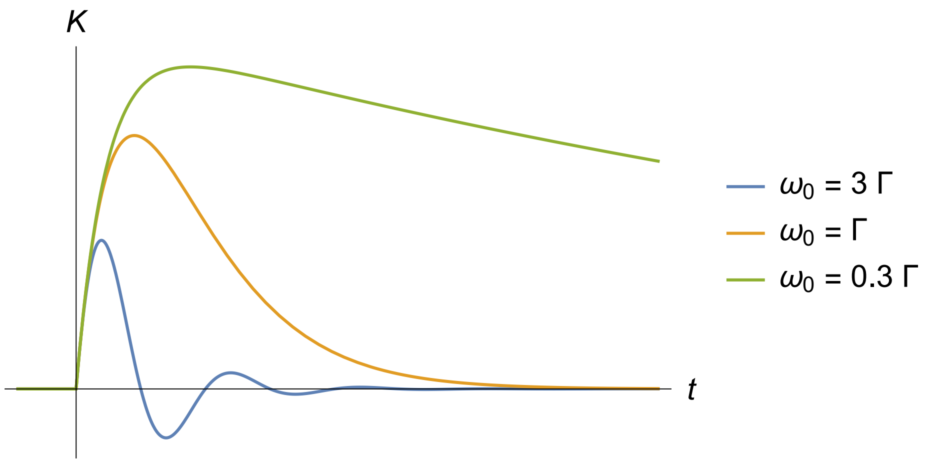
$\omega_0 > \Gamma \quad \to \quad \Omega$ is real (system is underdamped.)

$\omega_0 = \Gamma \quad \to \quad \Omega = 0, \; \frac{\sin \Omega\, t}{\Omega} = t$ (system is critically damped.)

$\omega_0 < \Gamma \quad \to \quad \Omega$ is imaginary (system is overdamped.)

The plots [see §Code] for these cases, with $m = 1$, are shown below.

Out[118]=



Answer (b)

Putting (8) into (10.160) gives

$$W_{abs} = \int_0^\infty \frac{d\,\omega}{\pi} \frac{\gamma\,\omega^2}{m^2\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\,\omega^2} F_0^2$$

$$= \frac{\gamma}{m\sqrt{2}\left(\sqrt{A + B - \gamma\sqrt{A}} + \sqrt{A + B + \gamma\sqrt{A}}\right)} F_0^2 \qquad \text{[See §Code.]}$$

where

$$A = \gamma^2 - 4\,m^2\,\omega_0^2 \qquad B = 2\,m^2\,\omega_0^2$$

Now,

$$\left(\sqrt{a} + \sqrt{b}\right)^2 = a + b + 2\sqrt{a\,b}$$

$$\to \quad \sqrt{a} + \sqrt{b} = \sqrt{a + b + 2\sqrt{a\,b}}$$

$$\therefore \quad \sqrt{A + B - \gamma\sqrt{A}} + \sqrt{A + B + \gamma\sqrt{A}} = \sqrt{2\,(A + B) + 2\sqrt{(A + B)^2 - \gamma^2\,A}}$$

$$= \sqrt{2}\sqrt{\left(\gamma^2 - 2\,m^2\,\omega_0^2 + \sqrt{\left(\gamma^2 - 2\,m^2\,\omega_0^2\right)^2 - \gamma^2\left(\gamma^2 - 4\,m^2\,\omega_0^2\right)}\right)}$$

$$= \sqrt{2}\sqrt{\gamma^2 - 2\,m^2\,\omega_0^2 + 2\,m^2\,\omega_0^2}$$

$$= \sqrt{2}\,\gamma$$

Hence,

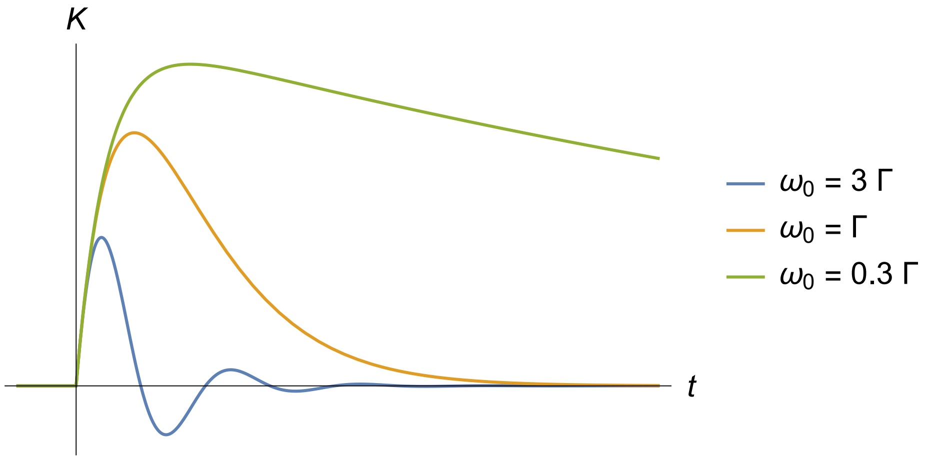
$$W_{abs} = \frac{F_0^2}{2\,m} \tag{8}$$

Code

In[131]:=
```
K[t_, Γ_, ω0_] := HeavisideTheta[t] (ⅇ^(-Γ t) Sin[Sqrt[ω0^2 - Γ^2] t]) / Sqrt[ω0^2 - Γ^2]
```

In[132]:=
```
Γ = 1;
pL = {K[t, Γ, 3 Γ], K[t, Γ, (1 + 10^-8) Γ], K[t, Γ, 0.3 Γ]};
Plot[pL // Evaluate, {t, -1, 10}, PlotRange → All,
    AxesLabel → {"t", "K"}, Ticks → None,
    PlotLegends → {"ω0 = 3 Γ", "ω0 = Γ", "ω0 = 0.3 Γ"}]
```

Out[134]=



In[136]:=
```
Assuming[m > 0 && γ > 0 && ω0 > 0, ∫_0^∞ (1/π) (γ ω^2)/(m^2 (ω0^2 - ω^2)^2 + γ^2 ω^2) ⅆω] // Simplify
```

Out[136]=
```
ConditionalExpression[γ / (Sqrt[2] m (Sqrt[γ^2 - 2 m^2 ω0^2 - γ Sqrt[γ^2 - 4 m^2 ω0^2]] + Sqrt[-2 m^2 ω0^2 + γ (γ + Sqrt[γ^2 - 4 m^2 ω0^2])])), 2 m ω0 ≤ γ]
```