Appendix A. Balance Equations

This section is from the 1st ed. of Reichl's text.

A.1.1. Convective Time Derivative

[For a more rigorous discussion, see chapter 1 of the book by R.E.Meyer, "Introduction to Mathematical Fluid Dynamics", Wiley (1971). Part I of the book "Fluid Mechanics", Springer (1997), by J.Spurk, gives a good introduction to the fundamentals of the subject.]

For our purposes, a fluid can be thought of as a collection of "fluid particles" that is dense enough to be considered as a continuum. Operationally, this means the characteristic lengths of the phenomena under study must be significantly larger than the average nearest neighbor particle distances. Here, a (fluid) particle means something that can be treated mathematically as occupying a geometric point in space but large enough physically so that fluctuations on the atomic scale can be neglected.

As a direct extension of the mechanics of discrete particles, the basic dynamical variables are the particle positions $\boldsymbol{x}(\boldsymbol{z}, t)$ and velocities $\boldsymbol{v}(\boldsymbol{z}, t)$, where $\boldsymbol{z}$ is the label of the particles. A property F of the system is then described by specifying the particle property $F(\boldsymbol{z}, t)$ for all $\boldsymbol{z}$ at each time t. This is called the Lagrangian formalism. Usually, one simply uses the positions of the particles at, say $t = 0$, to label them, i.e., $\boldsymbol{z} = \boldsymbol{x}(\boldsymbol{z}, 0)$. On the other hand, the natural way to describe a property F of a continuum is by means of a (field, or density) function $f(\boldsymbol{x}, t)$. This is called the Eulerian formalism.

We define the coordinate system that "moves" with the fluid as the one in which the position of any particle $\boldsymbol{z}$ is always $\boldsymbol{z}$. Note that the coordinate system "moves" (or, more properly speaking, adjust itself) with EVERY particle, even though they may move with different velocities. Obviously, this is not a coordinate system in the usual sense and cannot be actually constructed until the full dynamics of the fluid is known, i.e., after the problem is solved.

The Lagrangian and Eulerian formalisms are then equivalent to using a coordinate system that is moving with the fluid and fixed in space, respectively.

Now, $\boldsymbol{x}(t)$ denotes the trajectory of a particle that is in position $\boldsymbol{x}$ at time t. Its label is therefore $\boldsymbol{z} = \boldsymbol{x}(0)$ so that

$$\boldsymbol{x}(t) = \boldsymbol{x}(\boldsymbol{z}, t) = \boldsymbol{x}\left[\boldsymbol{x}(0), t\right] \tag{A.1}$$

The two formalisms are therefore related by

$$f(\boldsymbol{x}, t) = f\left[\boldsymbol{x}(\boldsymbol{z}, t), t\right] = F(\boldsymbol{z}, t) = F\left[\boldsymbol{x}(0), t\right] \tag{A.1a}$$

The convective time derivative is defined as

$$\frac{D}{Dt} f(\boldsymbol{x}, t) \equiv \left(\frac{\partial F(\boldsymbol{z}, t)}{\partial t}\right)_{\boldsymbol{z}} \tag{A.2a}$$

Since $\boldsymbol{z}$ is just the particle label, (A.2a) gives the change rate of F for particle $\boldsymbol{z}$, i.e., as seen in a frame moving with $\boldsymbol{z}$. Writing it in terms of f, we have

$$\begin{aligned} \frac{Df}{Dt} &= \left(\frac{\partial f\left[\boldsymbol{x}(\boldsymbol{z}, t), t\right]}{\partial t}\right)_{\boldsymbol{z}} \\ &= \left(\frac{\partial f(\boldsymbol{x}, t)}{\partial t}\right)_{\boldsymbol{x}} + \left(\frac{\partial \boldsymbol{x}(\boldsymbol{z}, t)}{\partial t}\right)_{\boldsymbol{z}} \cdot \left(\frac{\partial f(\boldsymbol{x}, t)}{\partial \boldsymbol{x}}\right)_{t} \end{aligned} \tag{A.2}$$

$$= \frac{\partial f(\boldsymbol{x},t)}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f(\boldsymbol{x},t)}{\partial \boldsymbol{x}} \tag{A.2b}$$

where

$$\boldsymbol{v} = \left(\frac{\partial \boldsymbol{x}(\boldsymbol{z},t)}{\partial t}\right)_{\boldsymbol{z}} = \frac{d\,\boldsymbol{x}(t)}{d\,t} = \text{velocity of particle } \boldsymbol{z}\text{, which is in position } \boldsymbol{x} \text{ at time } t.$$

Thus, the convective derivative is just the total time derivative

$$\frac{d}{d\,t} \equiv \frac{\partial}{\partial t} + \frac{d\,\boldsymbol{x}}{d\,t}\cdot\frac{\partial}{\partial \boldsymbol{x}}$$

$$= \frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} = \frac{D}{D\,t} \tag{A.4}$$

A.1.2. Volume Changes

Consider a given set of fluid particles occupying an infinitesimal volume $d\,V_{\boldsymbol{x}}(t) = d^3\,x(t)$, centered around the point $\boldsymbol{x}(t)$.

According to the definition (A.1), we have

$$d\,x_i(t) = \sum_j \frac{\partial x_i(\boldsymbol{z},t)}{\partial z_j}\,d\,z_j \qquad \boldsymbol{z} = \boldsymbol{x}(0)$$

$$= \sum_j \frac{\partial x_i(\boldsymbol{z},t)}{\partial z_j}\,d\,x_j(0)$$

Hence,

$$d\,V_{\boldsymbol{x}}(t) = d^3\,x(t) = J\,d\,V_{\boldsymbol{z}} = J\,d^3\,x(0) \tag{A.5}$$

where the Jacobian of the coordinate transformation is given by

$$J = \frac{\partial(x_1,x_2,x_3)}{\partial(z_1,z_2,z_3)} = \det\left|\frac{\partial x_i}{\partial z_j}\right| = \begin{vmatrix} \frac{\partial x_1}{\partial z_1} & \frac{\partial x_1}{\partial z_2} & \frac{\partial x_1}{\partial z_3} \\ \frac{\partial x_2}{\partial z_1} & \frac{\partial x_2}{\partial z_2} & \frac{\partial x_2}{\partial z_3} \\ \frac{\partial x_3}{\partial z_1} & \frac{\partial x_3}{\partial z_2} & \frac{\partial x_3}{\partial z_3} \end{vmatrix} \tag{A.8}$$

Now, as will be derived in §A.1.3, for a matrix $\boldsymbol{A} = \{a_{ij}\}$, its determinant $A = \det \boldsymbol{A}$ is given by the Laplace expansion

$$A\,\delta_i^j = \sum_k a_{ik}\,A^{jk}$$

where A^{jk} is the cofactor of a_{jk} defined as $(-)^{j+k}$ times the determinant of the sub-matrix obtained by striking out the j^{th} row & k^{th} column of $\boldsymbol{A}$. Hence,

$$\frac{\partial A}{\partial t} = \sum_{ij}\frac{\partial a_{ij}}{\partial t}\frac{\partial A}{\partial a_{ij}} = \sum_{ij}\frac{\partial a_{ij}}{\partial t}A^{ij}$$

so that (A.8) gives

$$\frac{\partial J}{\partial t} = \sum_{ij}\left(\frac{\partial}{\partial t}\frac{\partial x_i}{\partial z_j}\right)J^{ij} = \sum_{ij}\frac{\partial v_i}{\partial z_j}J^{ij} = \sum_{ijk}\frac{\partial v_i}{\partial x_k}\frac{\partial x_k}{\partial z_j}J^{ij}$$

$$= \sum_{ik}\frac{\partial v_i}{\partial x_k}\delta_k^i\,J = \sum_i \frac{\partial v_i}{\partial x_i}J$$

$$= J\,\nabla_{\boldsymbol{x}}\cdot\boldsymbol{v} \tag{A.12}$$

A.1.3. Determinants

Source: §7.3, R.D'Inverno, "Introducing Einstein's Relativity", Clarendon (92).

Consider a matrix $\boldsymbol{A} = \{a_{ij}\}$ with determinant $A = \det \boldsymbol{A}$.

The Laplace expansion of A is, with summation over repeated, staggered, indices implied,

$$A\,\delta^j_i = a_{ik}\,A^{jk} \qquad \text{(a)}$$

where the cofactor A^{ij} is defined as

$$A^{ij} = (-)^{i+j} \det \boldsymbol{\alpha}^{ij} \qquad \text{(b)}$$

where the (first) minor matrix $\boldsymbol{\alpha}^{ij}$ conjugate to element a_{ij} is obtained by striking out the i^{th} row and j^{th} column of $\boldsymbol{A}$.

The inverse $\boldsymbol{A}^{-1} = \{a^{ij}\}$ is given by

$$a^{ij} = \frac{1}{A} A^{ji}$$

so that

$$\boldsymbol{A}\,\boldsymbol{A}^{-1} = a_{ik}\,a^{kj} = \frac{1}{A} a_{ik}\,A^{jk} = \delta^j_i = \boldsymbol{I}$$

From (a), we have

$$\frac{\partial A}{\partial a_{ij}} = A^{ij}$$

so that

$$\frac{\partial A}{\partial t} = \frac{\partial A}{\partial a_{ij}} \frac{\partial a_{ij}}{\partial t} = A^{ij} \frac{\partial a_{ij}}{\partial t} \qquad \text{(c)}$$

A.2. General Balance Equation

Consider the integral

$$F(t) = \int_{V(t)} d^3 x\, f(\boldsymbol{x}, t) \qquad \text{(A.13)}$$

which can be interpreted as giving the total amount of a quantity F, of density $f(\boldsymbol{x}, t)$, for a fixed set of fluid particles occupying a volume $V(t)$ that moves with the fluid.

The rate of change of F in the frame moving with the fluid is given by the convective derivative

$$\frac{DF}{Dt} = \frac{D}{Dt} \int_{V(t)} d^3 x\, f(\boldsymbol{x}, t) \qquad \text{(A.14)}$$

$$= \frac{\partial}{\partial t} \int_{V_z} d^3 z\, J\, f(\boldsymbol{z}, t) \qquad \text{[(A.5) used.]}$$

$$= \int_{V_z} d^3 z\, \frac{\partial}{\partial t}\left[J\, f(\boldsymbol{z}, t)\right]$$

$$= \int_{V_z} d^3 z \left[\frac{\partial J}{\partial t} f(\boldsymbol{z}, t) + J \frac{\partial f(\boldsymbol{z}, t)}{\partial t}\right]$$

$$= \int_{V_z} d^3 z\, J\left[f(\boldsymbol{z}, t)\, \nabla_{\boldsymbol{x}} \cdot \boldsymbol{v} + \frac{\partial f(\boldsymbol{z}, t)}{\partial t}\right] \qquad \text{[(A.12) used.]}$$

$$= \int_{V(t)} d^3 x \left[f(\boldsymbol{x}, t)\, \nabla_{\boldsymbol{x}} \cdot \boldsymbol{v} + \frac{d f(\boldsymbol{x}, t)}{d t}\right] \qquad \text{(A.16)}$$

$$= \int_{V(t)} d^3 x \left[f(\boldsymbol{x}, t) \nabla_{\boldsymbol{x}} \cdot \boldsymbol{v} + \frac{\partial f(\boldsymbol{x}, t)}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} f(\boldsymbol{x}, t) \right]$$

$$= \int_{V(t)} d^3 x \left\{ \frac{\partial f(\boldsymbol{x}, t)}{\partial t} + \nabla_{\boldsymbol{x}} \cdot \left[\boldsymbol{v} f(\boldsymbol{x}, t) \right] \right\}$$

Now, the change in F can be caused by a source with density generating rate σ_F, or by a flux $\boldsymbol{J}_F$ so that

$$\frac{D F}{D t} = \int_{V(t)} d^3 x \, \sigma_F - \oint_{\boldsymbol{S}(t)} d\, \boldsymbol{S} \cdot \boldsymbol{J}_F \qquad \text{(A.17)}$$

$$= \int_{V(t)} d^3 x \left(\sigma_F - \nabla_{\boldsymbol{x}} \cdot \boldsymbol{J}_F \right)$$

where $\boldsymbol{S}(t)$ is the surface bounding $V(t)$ and we've used the Green's theorem.
Since (A.16,17) are valid for arbitrary $V(t)$, we must have

$$f \nabla_{\boldsymbol{x}} \cdot \boldsymbol{v} + \frac{D f}{D t} = \sigma_F - \nabla_{\boldsymbol{x}} \cdot \boldsymbol{J}_F$$

$$\to \qquad \frac{D f}{D t} = -f \nabla_{\boldsymbol{x}} \cdot \boldsymbol{v} + \sigma_F - \nabla_{\boldsymbol{x}} \cdot \boldsymbol{J}_F \qquad \text{(A.18)}$$

which is known as the balance equation.

Using

$$\frac{D f}{D t} = \frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} f$$

we have

$$\frac{\partial f}{\partial t} + \nabla_{\boldsymbol{x}} \cdot (\boldsymbol{v} f) = \sigma_F - \nabla_{\boldsymbol{x}} \cdot \boldsymbol{J}_F$$

$$\frac{\partial f}{\partial t} + \nabla_{\boldsymbol{x}} \cdot (\boldsymbol{J}_C + \boldsymbol{J}_F) = \sigma_F \qquad \text{(A.19)}$$

where $\boldsymbol{J}_C = \boldsymbol{v} f$ is the convective current density of F.

Integrating (A.19) over a volume V_f fixed in space, we have

$$\int_{V_f} d^3 x \frac{\partial f}{\partial t} = -\oint_{\boldsymbol{S}_f} d\, \boldsymbol{S} \cdot (\boldsymbol{J}_C + \boldsymbol{J}_F) + \int_{V_f} d^3 x \, \sigma_F \qquad \text{(A.20)}$$

$$= \frac{d}{d t} \int_{V_f} d^3 x \, f \qquad \text{(A.21)}$$