Time Reversal Symmetry

The time reversal operator $\mathcal{T}$ reverses the direction of time, i.e., it is a reflection operator on the t-space

$$\mathcal{T}\, t = \tilde{t} = -t \qquad (1)$$

with inverse

$$\mathcal{T}^{-1} = \mathcal{T} \qquad (1a)$$

According to the theory of operators and transformations, we have, for any function $f(t)$,

$$\mathcal{T} f = \tilde{f}$$

such that

$$\mathcal{T} f(\mathcal{T}\, t) = \tilde{f}\,(\tilde{t}) = f(t) \qquad (2)$$

$$\rightarrow \quad \tilde{f}\,(t) = f\left(\mathcal{T}^{-1}\, t\right) = f\,(\tilde{t}) = f(-t) \qquad (2a)$$

Consider now the effects of $\mathcal{T}$ on a system of coordinate $\boldsymbol{q}(t)$, velocity $\dot{\boldsymbol{q}}(t)$, momentum $\boldsymbol{p}(t)$, Lagrangian L, Hamiltonian H, and action S.

To begin,

$$\mathcal{T}\boldsymbol{q}(t) = \tilde{\boldsymbol{q}}\,(t) = \boldsymbol{q}(-t) \qquad (3)$$

Using

$$\dot{\boldsymbol{q}}\,(t) = \frac{d\,\boldsymbol{q}(t)}{d\,t} = \lim_{t'\to t} \frac{\boldsymbol{q}(t') - \boldsymbol{q}(t)}{t' - t}$$

$$\dot{\tilde{\boldsymbol{q}}}\,(t) = \frac{d\,\tilde{\boldsymbol{q}}(t)}{d\,t} = \lim_{t'\to t} \frac{\tilde{\boldsymbol{q}}(t') - \tilde{\boldsymbol{q}}(t)}{t' - t}$$

we have

$$\mathcal{T}\,\dot{\boldsymbol{q}}(t) \equiv \tilde{\dot{\boldsymbol{q}}}(t) = \lim_{t'\to t} \mathcal{T}\left(\frac{\boldsymbol{q}(t') - \boldsymbol{q}(t)}{t' - t}\right) = \lim_{t'\to t} \frac{\tilde{\boldsymbol{q}}(t') - \tilde{\boldsymbol{q}}(t)}{-t' + t} = -\dot{\tilde{\boldsymbol{q}}}(t)$$

$$= \lim_{t'\to t} \frac{\boldsymbol{q}(-t') - \boldsymbol{q}(-t)}{-t' + t} = \dot{\boldsymbol{q}}(-t) \qquad (4)$$

in agreement with (2a).

Thus,

$$\mathcal{T}\,\dot{\boldsymbol{q}} \equiv \tilde{\dot{\boldsymbol{q}}} = -\dot{\tilde{\boldsymbol{q}}}$$

$$\rightarrow \quad \mathcal{T}\frac{d}{d\,t} = -\frac{d}{d\,t}\mathcal{T} \qquad (4a)$$

(4) can also be derived as

$$\mathcal{T}\,\dot{\boldsymbol{q}}(t) = \mathcal{T}\left(\frac{d\,\boldsymbol{q}(t)}{d\,t}\right) = \frac{d\,\tilde{\boldsymbol{q}}(t)}{d\,\tilde{t}} = -\frac{d\,\tilde{\boldsymbol{q}}(t)}{d\,t} = -\dot{\tilde{\boldsymbol{q}}}(t)$$

$$= \frac{d\,\boldsymbol{q}(\tilde{t})}{d\,\tilde{t}} = \dot{\boldsymbol{q}}(\tilde{t}) = \dot{\boldsymbol{q}}(-t)$$

where we've used

$$\frac{d\,f(x_0)}{d\,x} \equiv \left.\frac{d\,f(x)}{d\,x}\right|_{x=x_0} \equiv \lim_{x\to x_0} \frac{f\,(x) - f(x_0)}{x - x_0}$$

= derivative of f with respect to x evaluated at $x = x_0$.

For a time-dependent Lagrangian, t plays a double role under $\mathcal{T}$. This can be clarified by introducing the identity function $\tau(t) = t$, as follows.

If the Lagrangian is time-independent,

$$L = L(\boldsymbol{q}, \dot{\boldsymbol{q}})$$

Its value at t is given by

$$L(t) = L[\boldsymbol{q}(t),\ \dot{\boldsymbol{q}}(t)]$$

If the Lagrangian is explicitly dependent on t,

$$L = L(\boldsymbol{q},\ \dot{\boldsymbol{q}},\ \tau) \tag{5}$$

Its value at t is given by

$$L(t) = L[\boldsymbol{q}(t),\ \dot{\boldsymbol{q}}(t),\ \tau(t)] = L[\boldsymbol{q}(t),\ \dot{\boldsymbol{q}}(t),\ t] \tag{5a}$$

For example, a 1-D oscillator driven by a sinusoidal force is described by

$$L(x,\ \dot{x},\ \tau) = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2 + F x \sin \Omega \tau \tag{6}$$

$$\begin{aligned} L(t) &= L[x(t),\ \dot{x}(t),\ \tau(t)] \\ &= \frac{1}{2} m \dot{x}(t)^2 - \frac{1}{2} m \omega^2 x(t)^2 + F x(t) \sin \Omega t \end{aligned} \tag{6a}$$

Thus,

$$\mathcal{T} L = \tilde{L} \equiv L(\mathcal{T}\boldsymbol{q}, \mathcal{T}\dot{\boldsymbol{q}}, \mathcal{T}\tau) = L\left(\tilde{\boldsymbol{q}},\ \tilde{\dot{\boldsymbol{q}}},\ \tilde{\tau}\right) = L\left(\tilde{\boldsymbol{q}},\ -\dot{\tilde{\boldsymbol{q}}},\ \tilde{\tau}\right) \tag{7}$$

where

$$\mathcal{T}\tau(t) = \tilde{\tau}(t) = \tau(-t) = -t \tag{7a}$$

so that

$$\begin{aligned} \mathcal{T} L(t) = \tilde{L}(t) = L[\mathcal{T}\boldsymbol{q}(t), \mathcal{T}\dot{\boldsymbol{q}}(t), \mathcal{T}\tau(t)] &= L\left[\tilde{\boldsymbol{q}}(t),\ \tilde{\dot{\boldsymbol{q}}}(t),\ \tilde{\tau}(t)\right] \\ = L[\boldsymbol{q}(-t),\ \dot{\boldsymbol{q}}(-t),\ -t] &= L(-t) \end{aligned} \tag{7b}$$

in agreement with (2a).

Consider the action

$$S = \int_{t_0}^{t_1} d t\ L(t) = \int_{t_0}^{t_1} d t\ L[\boldsymbol{q}(t),\ \dot{\boldsymbol{q}}(t),\ \tau(t)] \tag{8}$$

Using [see (2)]

$$\tilde{\boldsymbol{q}}(\tilde{t}) = \boldsymbol{q}(t) \qquad \tilde{\dot{\boldsymbol{q}}}(\tilde{t}) = \dot{\boldsymbol{q}}(t) \qquad \tilde{\tau}(\tilde{t}) = \tau(t)$$

(8) becomes

$$S = -\int_{-t_0}^{-t_1} d\tilde{t}\ L(\tilde{t}) = -\int_{-t_0}^{-t_1} d\tilde{t}\ L\left[\tilde{\boldsymbol{q}}(\tilde{t}),\ \tilde{\dot{\boldsymbol{q}}}(\tilde{t}),\ \tilde{\tau}(\tilde{t})\right] \tag{8a}$$

On the other hand,

$$\begin{aligned} \mathcal{T} S &= \mathcal{T}\left[\int_{t_0}^{t_1} d t\ L(t)\right] \\ &= \int_{-t_0}^{-t_1} d\tilde{t}\ \tilde{L}(t) \\ &= -S \qquad \text{[(8a) used.]} \end{aligned}$$

Since S & $-S$ are extremized by the same trajectories, the dynamics of the system can be equally described by L or $\tilde{L}$.

In other words, all Lagrangian dynamics are time reversal invariant, even though L may not be.

Note: dissipative systems, such as the dissipative oscillator, are obviously not time reversal invariant. This is acceptable since they cannot be described by a Lagrangian.

The variation $\delta S = 0$ on (8) & (8a) then gives two equivalent Euler-Lagrange equations

$$\frac{d}{d t} \frac{\partial L(t)}{\partial \dot{\boldsymbol{q}}} - \frac{\partial L(t)}{\partial \boldsymbol{q}} = 0 \tag{9}$$

$$\frac{d}{d\tilde{t}}\frac{\partial L(\tilde{t})}{\partial \dot{\tilde{\boldsymbol{q}}}} - \frac{\partial L(\tilde{t})}{\partial \tilde{\boldsymbol{q}}} = 0 \tag{9a}$$

where [see (4)]

$$\dot{\tilde{\boldsymbol{q}}}(\tilde{t}) = \frac{d\,\tilde{\boldsymbol{q}}(\tilde{t})}{d\,\tilde{t}} \tag{9b}$$

(9a) can be written as

$$\frac{d}{d(-t)}\frac{\partial L(-t)}{\partial\left(-\dot{\tilde{\boldsymbol{q}}}\right)} - \frac{\partial L(-t)}{\partial \tilde{\boldsymbol{q}}} = 0$$

$$\rightarrow \qquad \frac{d}{dt}\frac{\partial L(t)}{\partial \dot{\tilde{\boldsymbol{q}}}} - \frac{\partial L(t)}{\partial \tilde{\boldsymbol{q}}} = 0 \qquad (-t \rightarrow t) \tag{9c}$$

where

$$L(t) = L\left[\tilde{\boldsymbol{q}}(t),\ \dot{\tilde{\boldsymbol{q}}}(t),\ \tilde{\tau}(t)\right] = L\left[\tilde{\boldsymbol{q}}(t),\ -\dot{\tilde{\boldsymbol{q}}}(t),\ \tilde{\tau}(t)\right]$$

Hence, the Lagrangian for $\tilde{\boldsymbol{q}}(t)$ has the same form as that for $\boldsymbol{q}(t)$ but with the velocity reversed in direction.

Thus, if $\boldsymbol{q}(t)$ is an actual trajectory of the system that

passes through $\quad \boldsymbol{q}(t_0) = \boldsymbol{q}_0 \quad$ with velocity $\quad \dot{\boldsymbol{q}}(t_0) = \dot{\boldsymbol{q}}_0$

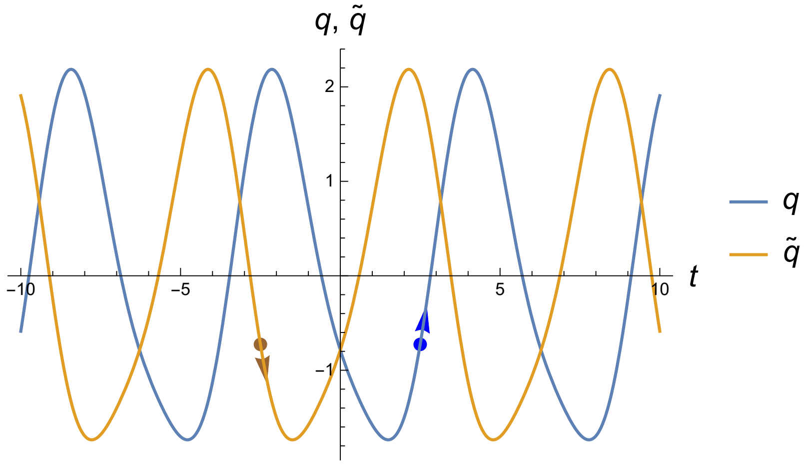
then $\tilde{\boldsymbol{q}}(t) = \boldsymbol{q}(-t)$ is the same spatial trajectory traveled backwards in time and

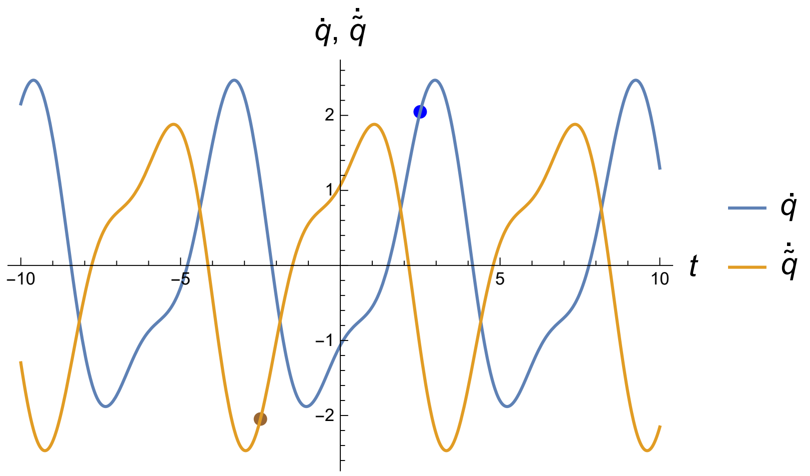
passes through $\quad \tilde{\boldsymbol{q}}(-t_0) = \boldsymbol{q}_0 \quad$ with velocity $\quad \dot{\tilde{\boldsymbol{q}}}(-t_0) = -\dot{\boldsymbol{q}}_0 \qquad$ (10)

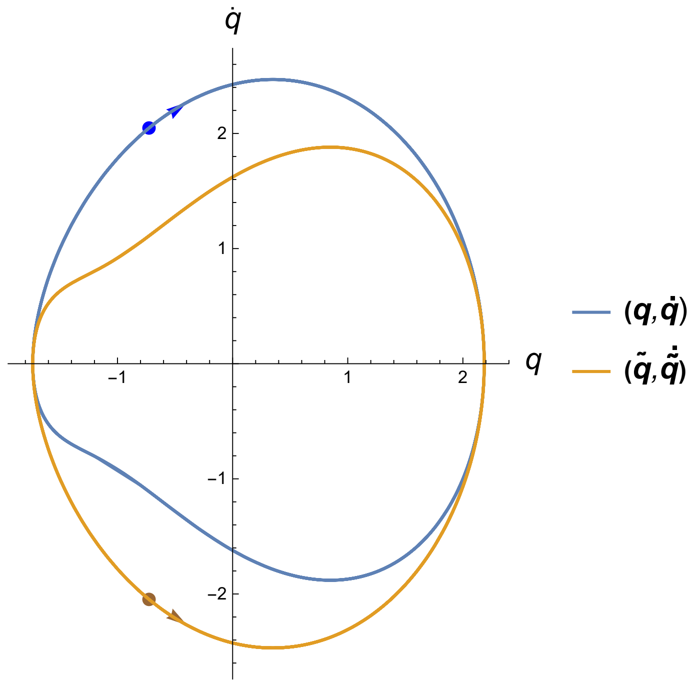
Obviously, (4) still holds so that

$$\dot{\tilde{\boldsymbol{q}}}(t) = -\dot{\boldsymbol{q}}(-t) \qquad \forall\ t$$

For example, the Lagrangian in (6) gives [see §Code], we have







where the dots are related by

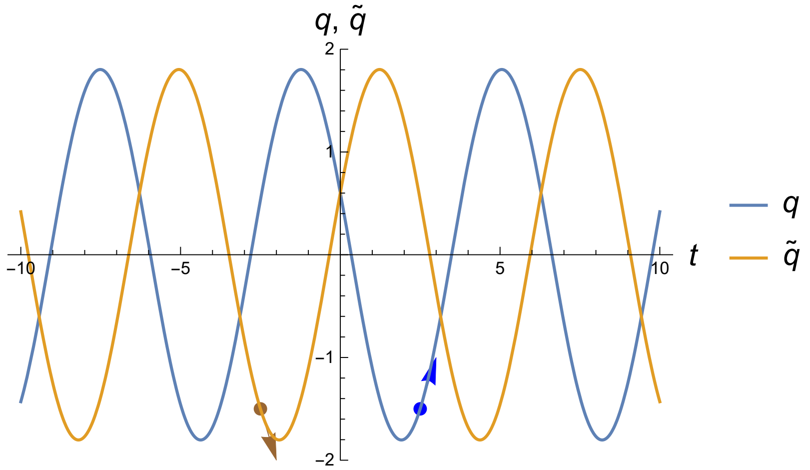
$$\tilde{q}(-t_0) = q(t_0) \qquad \text{or} \qquad \dot{\tilde{q}}(-t_0) = -\dot{q}(t_0)$$

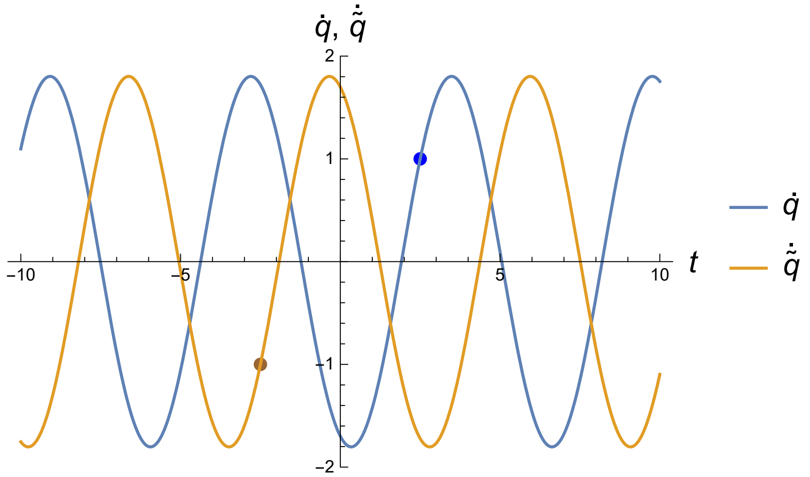
Note that a system point on the trajectories $(q, \dot{q})$ & $\left(\tilde{q}, \dot{\tilde{q}}\right)$ in phase space evolves in the clockwise & counterclockwise sense, respectively.

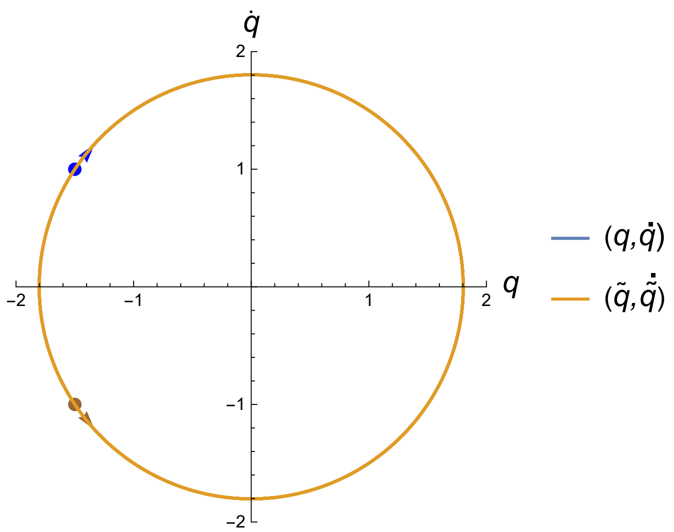
If we take out the sine term in (6) so that

$$L = L(x, \dot{x}) = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2$$

then L itself is time-reversal invariant. i.e., $\tilde{L}(t) = L(-t) = L(t)$. The solutions are as follows:







The only difference from the time-dependent case is that trajectories $(q, \dot{q})$ & $\left(\tilde{q}, \dot{\tilde{q}}\right)$ in phase space coincide. This is a characteristic feature for time reversal invariant Lagrangians.

Using

$$\boldsymbol{p} \equiv \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \qquad \tilde{\boldsymbol{p}}_L \equiv \frac{\partial \tilde{L}}{\partial \dot{\tilde{\boldsymbol{q}}}} = \frac{\partial \tilde{L}\left(\tilde{\boldsymbol{q}}, \dot{\tilde{\boldsymbol{q}}}, \tilde{\tau}\right)}{\partial \dot{\tilde{\boldsymbol{q}}}} \tag{11}$$

we have

$$\mathcal{T}\boldsymbol{p} \equiv \tilde{\boldsymbol{p}} = \mathcal{T}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right) = -\frac{\partial \tilde{L}}{\partial \dot{\tilde{\boldsymbol{q}}}} \tag{11a}$$

$$\mathcal{T}\boldsymbol{p}(t) = \tilde{\boldsymbol{p}}(t) = -\frac{\partial \tilde{L}(t)}{\partial \dot{\tilde{\boldsymbol{q}}}(t)} = \frac{\partial L(-t)}{\partial \dot{\boldsymbol{q}}(-t)} = \boldsymbol{p}(-t) \tag{11b}$$

$$\tilde{\boldsymbol{p}}_L(t) = \frac{\partial \tilde{L}(t)}{\partial \dot{\tilde{\boldsymbol{q}}}(t)} = \frac{\partial L(-t)}{-\partial \dot{\boldsymbol{q}}(-t)} = -\boldsymbol{p}(-t) = -\tilde{\boldsymbol{p}}(t) \tag{11c}$$

which should be compared with (4).

For the Hamiltonian

$$H \equiv \dot{\boldsymbol{q}} \cdot \frac{\partial L}{\partial \dot{\boldsymbol{q}}} - L = \dot{\boldsymbol{q}} \cdot \boldsymbol{p} - L = H(\boldsymbol{q}, \boldsymbol{p}, \tau) \tag{12}$$

we have

$$\mathcal{T} H = \tilde{H} = H(\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}, \tilde{\tau}) = \tilde{\dot{\boldsymbol{q}}} \cdot \tilde{\boldsymbol{p}} - \tilde{L} = -\dot{\tilde{\boldsymbol{q}}} \cdot \tilde{\boldsymbol{p}} - \tilde{L} \tag{12a}$$

$$\to \quad \mathcal{T} H(t) = \tilde{H}(t) = \dot{\boldsymbol{q}}(-t) \cdot \boldsymbol{p}(-t) - L(-t) = H(-t) \tag{12b}$$

Applying $\mathcal{T}$ to the Hamilton's equations

$$\dot{\boldsymbol{q}} = \frac{\partial H}{\partial \boldsymbol{p}} \qquad \dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{q}} \tag{13}$$

we have

$$\mathcal{T}\dot{\boldsymbol{q}} = \tilde{\dot{\boldsymbol{q}}} = -\dot{\tilde{\boldsymbol{q}}} = \frac{\partial \tilde{H}}{\partial \tilde{\boldsymbol{p}}} = -\frac{\partial \tilde{H}}{\partial \tilde{\boldsymbol{p}}_L} \qquad \mathcal{T}\dot{\boldsymbol{p}} = \tilde{\dot{\boldsymbol{p}}} = -\dot{\tilde{\boldsymbol{p}}} = \dot{\tilde{\boldsymbol{p}}}_L = -\frac{\partial \tilde{H}}{\partial \tilde{\boldsymbol{q}}}$$

Since the canonical variables of $\tilde{H}$ are $\tilde{\boldsymbol{q}}$ & $\tilde{\boldsymbol{p}}_L$, the corresponding Hamilton's equations are

$$\dot{\tilde{\boldsymbol{q}}} = \frac{\partial \tilde{H}}{\partial \tilde{\boldsymbol{p}}_L} \qquad \dot{\tilde{\boldsymbol{p}}}_L = -\frac{\partial \tilde{H}}{\partial \tilde{\boldsymbol{q}}} \tag{13a}$$

where, from (12a),

$$\tilde{H} = -\dot{\tilde{\boldsymbol{q}}} \cdot \tilde{\boldsymbol{p}} - \tilde{L} = \dot{\tilde{\boldsymbol{q}}} \cdot \tilde{\boldsymbol{p}}_L - \tilde{L} = \tilde{H}(\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}_L, \tilde{\tau}) = H(\tilde{\boldsymbol{q}}, -\tilde{\boldsymbol{p}}_L, \tilde{\tau}) \tag{13b}$$

Hence, given a solution to (13)

$$\{\boldsymbol{q}(t), \boldsymbol{p}(t)\} \quad \text{with} \quad \{\boldsymbol{q}(t_0), \boldsymbol{p}(t_0)\} = \{\boldsymbol{q}_0, \boldsymbol{p}_0\}$$

its time reversal

$$\{\tilde{\boldsymbol{q}}(t), \tilde{\boldsymbol{p}}_L(t)\} = \{\boldsymbol{q}(-t), -\boldsymbol{p}(-t)\}$$

is a solution to (13a) with

$$\{\tilde{\boldsymbol{q}}(-t_0), \tilde{\boldsymbol{p}}(-t_0)\} = \{\boldsymbol{q}_0, -\boldsymbol{p}_0\} \tag{13c}$$

The Hamiltonian version of the results obtained for the Lagrangian given in (6) are easily obtained since $p = m\dot{q}$.

Code

In[1]:=
```
(* Euler-Lagrange eq. *)
ELeq[L_, q_, t_] := ∂t (∂q'[t] L) - ∂q[t] L == 0
```

In[2]:=
```
(* Hamiltionian *)
HL[L_, q_, p_, t_] := Module[{sol},
  sol = Solve[p[t] == ∂q'[t] L, q'[t]];
  q'[t] ∂q'[t] L - L /. sol[[1]]
 ]
```

In[3]:=
```
(* Hamilton's eq. *)
Heq[H_, q_, p_, t_] := {q'[t] == ∂p[t] H, p'[t] == -∂q[t] H}
```

In[4]:=
```
L = 1/2 m q'[t]^2 - 1/2 m ω^2 q[t]^2 - F q[t] Sin[Ω t]
```

Out[4]= $-\frac{1}{2} m \omega^2 q[t]^2 - F\, q[t]\, \mathrm{Sin}[t\,\Omega] + \frac{1}{2} m\, q'[t]^2$

In[5]:=
```
eqL = ELeq[L, q, t]
```

Out[5]= $m\,\omega^2 q[t] + F\,\mathrm{Sin}[t\,\Omega] + m\,q''[t] == 0$

In[6]:=
```
H = HL[L, q, p, t]
```

Out[6]= $\frac{p[t]^2}{2m} + \frac{1}{2} m\,\omega^2 q[t]^2 + F\,q[t]\,\mathrm{Sin}[t\,\Omega]$

In[7]:=
```
eqH = Heq[H, q, p, t]
```

Out[7]= $\left\{q'[t] == \frac{p[t]}{m},\ p'[t] == -m\,\omega^2 q[t] - F\,\mathrm{Sin}[t\,\Omega]\right\}$

In[53]:=
```
par = {m → 1, F → 1, ω → 1, Ω → 2, q0 → -1.5, qp0 → 1, p0 → 1, t0 → 2.5, t1 → 2.65, Δt → .2};
```

In[35]:=
```
qs[t_] =
  q[t] /. (DSolve[{eqL, q[t0] == q0, q'[t0] == qp0} /. par // Evaluate, q, t] // Flatten)
```

Out[35]= $0.666667\,(-1.17944\,\mathrm{Cos}[t] - 2.59807\,\mathrm{Sin}[t] + 0.75\,\mathrm{Cos}[t]\,\mathrm{Sin}[t] + \mathrm{Cos}[t]^3\,\mathrm{Sin}[t] - 0.25\,\mathrm{Cos}[t]\,\mathrm{Sin}[3\,t])$

In[10]:=
```
qsT[t_] := qs[-t]
```

In[40]:=
```
Plot[{qs[t], qsT[t]}, {t, -10, 10},
 AxesLabel → {"t", "q, q̃"},
 PlotLegends → {"q", "q̃"},
 Prolog → {PointSize[Large], Blue, Point[{t0, qs[t0]} /. par],
   Arrow[{{t0, qs[t0]}, {t0 + Δt, qs[t0] + qs'[t0] Δt}} /. par],
   Brown, Point[{-t0, qsT[-t0]} /. par],
   Arrow[{{-t0, qsT[-t0]}, {-t0 + Δt, qsT[-t0] + qsT'[-t0] Δt}} /. par]}]
```

In[41]:=
```
Plot[{qs'[t], qsT'[t]}, {t, -10, 10},
 AxesLabel → {"t", "q̇, q̃̇"},
 PlotLegends → {"q̇", "q̃̇"}, Prolog → {PointSize[Large], Blue,
   Point[{t0, qs'[t0]} /. par], Brown, Point[{-t0, qsT'[-t0]} /. par]}]
```

In[54]:=
```
lst0 = {qs[t], qs'[t]} /. par;
lst1 = {qsT[t], qsT'[t]} /. par;
ParametricPlot[{lst0, lst1} // Evaluate, {t, -10, 10},
 AxesLabel → {"q", "q̇"}, PlotLegends → {"(q,q̇)", "(q̃,q̃̇)"},
 Prolog → {PointSize[Large], Blue, Point[{qs[t0], qs'[t0]} /. par],
   Arrow[{{qs[t0], qs'[t0]}, {qs[t1], qs'[t1]}} /. par],
   Brown, Point[{qsT[-t0], qsT'[-t0]} /. par],
   Arrow[{{qsT[-t0], qsT'[-t0]}, {qsT[-t1], qsT'[-t1]}} /. par]}]
```

Code I

In[69]:=
```
L = 1/2 m q'[t]^2 - 1/2 m ω^2 q[t]^2
```
Out[69]= $-\frac{1}{2} m\,\omega^2 q[t]^2 + \frac{1}{2} m\,q'[t]^2$

In[70]:=
```
eqL = ELeq[L, q, t]
```
Out[70]= $m\,\omega^2 q[t] + m\,q''[t] == 0$

In[71]:=
```
H = HL[L, q, p, t]
```
Out[71]= $\frac{p[t]^2}{2 m} + \frac{1}{2} m\,\omega^2 q[t]^2$

In[72]:=
```
eqH = Heq[H, q, p, t]
```
Out[72]= $\left\{q'[t] == \frac{p[t]}{m},\ p'[t] == -m\,\omega^2 q[t]\right\}$

In[73]:=
```
par = {m → 1, F → 1, ω → 1, Ω → 2, q0 → -1.5, qp0 → 1, p0 → 1, t0 → 2.5, t1 → 2.65, Δt → .5};
```

In[74]:=
```
qs[t_] =
  q[t] /. (DSolve[{eqL, q[t0] == q0, q'[t0] == qp0} /. par // Evaluate, q, t] // Flatten)
```
Out[74]= 0.603243 Cos[t] - 1.69885 Sin[t]

In[75]:=
```
qsT[t_] := qs[-t]
```

In[76]:=
```
Plot[{qs[t], qsT[t]}, {t, -10, 10},
 AxesLabel → {"t", "q, q̃"},
 PlotLegends → {"q", "q̃"},
 Prolog → {PointSize[Large], Blue, Point[{t0, qs[t0]} /. par],
   Arrow[{{t0, qs[t0]}, {t0 + Δt, qs[t0] + qs'[t0] Δt}} /. par],
   Brown, Point[{-t0, qsT[-t0]} /. par],
   Arrow[{{-t0, qsT[-t0]}, {-t0 + Δt, qsT[-t0] + qsT'[-t0] Δt}} /. par]}]
```

In[77]:=
```
Plot[{qs'[t], qsT'[t]}, {t, -10, 10},
 AxesLabel → {"t", "q̇, q̃̇"},
 PlotLegends → {"q̇", "q̃̇"}, Prolog → {PointSize[Large], Blue,
   Point[{t0, qs'[t0]} /. par], Brown, Point[{-t0, qsT'[-t0]} /. par]}]
```

In[78]:=
```
lst0 = {qs[t], qs'[t]} /. par;
lst1 = {qsT[t], qsT'[t]} /. par;
ParametricPlot[{lst0, lst1} // Evaluate, {t, -10, 10},
 AxesLabel → {"q", "q̇"}, PlotLegends → {"(q,q̇)", "(q̃,q̃̇)"},
 Prolog → {PointSize[Large], Blue, Point[{qs[t0], qs'[t0]} /. par],
   Arrow[{{qs[t0], qs'[t0]}, {qs[t1], qs'[t1]}} /. par],
   Brown, Point[{qsT[-t0], qsT'[-t0]} /. par],
   Arrow[{{qsT[-t0], qsT'[-t0]}, {qsT[-t1], qsT'[-t1]}} /. par]}]
```