11.C.1.1. Dynamics of the Scattering Process

Mathematica code for all graphics in this section can be found in file "scattering.nb".

Consider two particles with masses m_j, positions $\boldsymbol{r}_j$, momenta $\boldsymbol{p}_j$ and interacting via a short-range central potential $V(\,|\,\boldsymbol{r}_1 - \boldsymbol{r}_2\,|\,)$ in the laboratory (lab) frame. As usual, we set

$M \equiv m_1 + m_2$ = total mass.

$\mu \equiv \dfrac{m_1\, m_2}{M}$ = reduced mass.

$\boldsymbol{R} \equiv \dfrac{1}{M}(m_1\, \boldsymbol{r}_1 + m_2\, \boldsymbol{r}_2)$ = center of mass (CM) position vector.

$\boldsymbol{r} \equiv \boldsymbol{r}_2 - \boldsymbol{r}_1$ = relative position vector.

$$\rightarrow \qquad \boldsymbol{r}_1 = \boldsymbol{R} - \frac{m_2}{M}\boldsymbol{r} \qquad\qquad \boldsymbol{r}_2 = \boldsymbol{R} + \frac{m_1}{M}\boldsymbol{r}$$

(11.56)

$$\boldsymbol{p}_1 = m_1\, \dot{\boldsymbol{r}}_1 = m_1\, \boldsymbol{v}_1 \qquad\qquad \boldsymbol{p}_2 = m_2\, \dot{\boldsymbol{r}}_2 = m_2\, \boldsymbol{v}_2$$

Let

$\boldsymbol{P} \equiv M\,\dot{\boldsymbol{R}} = M\,\boldsymbol{V}_{CM} = \boldsymbol{p}_1 + \boldsymbol{p}_2$ = total momentum.

(11.56a)

$\boldsymbol{p} \equiv \mu\,\dot{\boldsymbol{r}} = \mu\,\boldsymbol{v} = \dfrac{m_1\, m_2}{M}(\boldsymbol{v}_2 - \boldsymbol{v}_1) = \dfrac{1}{M}(m_1\,\boldsymbol{p}_2 - m_2\,\boldsymbol{p}_1)$ = relative momentum.

then

$$\frac{1}{2M}P^2 + \frac{1}{2\mu}p^2 = \frac{1}{2M}(\boldsymbol{p}_1 + \boldsymbol{p}_2)^2 + \frac{1}{2\, m_1\, m_2\, M}(m_1\,\boldsymbol{p}_2 - m_2\,\boldsymbol{p}_1)^2$$
$$= \frac{1}{2M}\left(p_1^2 + p_2^2 + 2\,\boldsymbol{p}_1\cdot\boldsymbol{p}_2 + \frac{m_1}{m_2}p_2^2 + \frac{m_2}{m_1}p_1^2 - 2\,\boldsymbol{p}_2\cdot\boldsymbol{p}_1\right)$$
$$= \frac{1}{2\, m_1}p_1^2 + \frac{1}{2\, m_2}p_2^2$$

(11.56b)

The total Hamiltonian in the lab frame is

$$H_{tot} = \frac{1}{2\, m_1}p_1^2 + \frac{1}{2\, m_2}p_2^2 + V(\,|\,\boldsymbol{r}_1 - \boldsymbol{r}_2\,|\,)$$
$$= \frac{1}{2M}P^2 + \frac{1}{2\mu}p^2 + V(r) \qquad\qquad [\text{ (11.56b) used. }]$$
$$= E_{tot}$$

(11.57)

Thus,

$$H_{tot} = H_{CM} + H$$

where

$$H_{CM} = \frac{1}{2M}P^2 = E_{CM}$$

$$H = \frac{1}{2\mu} p^2 + V(r) = E = E_{\text{tot}} - E_{\text{CM}}$$
(11.57a)

The corresponding Hamilton's equations are

$$\dot{\boldsymbol{R}} = \boldsymbol{V}_{\text{CM}} = \frac{\partial H_{\text{tot}}}{\partial \boldsymbol{P}} = \frac{\partial H_{\text{CM}}}{\partial \boldsymbol{P}} = \frac{1}{M} \boldsymbol{P}$$ [In agreement with (11.56a).]

$$\dot{\boldsymbol{P}} = -\frac{\partial H_{\text{tot}}}{\partial \boldsymbol{R}} = -\frac{\partial H_{\text{CM}}}{\partial \boldsymbol{R}} = 0 \qquad \to \qquad \boldsymbol{P} = \text{const}$$
(11.57b)

[CM moves with a constant velocity.]

$$\dot{\boldsymbol{r}} = \frac{\partial H_{\text{tot}}}{\partial \boldsymbol{p}} = \frac{\partial H}{\partial \boldsymbol{p}} = \frac{1}{\mu} \boldsymbol{p}$$ [In agreement with (11.56a).]

$$\dot{\boldsymbol{p}} = -\frac{\partial H_{\text{tot}}}{\partial \boldsymbol{r}} = -\frac{\partial H}{\partial \boldsymbol{r}} = -\frac{\partial V(r)}{\partial \boldsymbol{r}}$$
(11.57c)

Since $\boldsymbol{V}_{\text{CM}} = \text{const}$, the frame attached to the center of mass, called the CM frame for short, is an inertial frame.

(11.57c) describes the motion of a particle of mass μ scattering off a potential $V(r)$ centered at the coordinate origin. However, its position $\boldsymbol{r}(t)$ and velocity $\boldsymbol{v}(t)$ are exactly the same as the relative coordinates & velocity, respectively, of our scattering problem in the lab frame. (11.57c) is therefore often called the equation of motion in the lab frame.

Note that although $\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1$, (11.57c) is not the equation of motion of particle 2 in a frame attached to particle 1. This is because $\boldsymbol{v}_1$ is not a constant and a frame attach to it is not inertial.

We shall use the subscripts 0 & f to denote initial & final values of a quantity, respectively.

In a typical scattering situation, two particles start out separated by a distance greater than the range r_c of $V(r)$. The initial total energy is therefore purely kinetic:

$$E_{\text{tot},0} = \frac{1}{2 m_1} p_{10}^2 + \frac{1}{2 m_2} p_{20}^2 = \frac{1}{2M} P^2 + \frac{1}{2\mu} p_0^2$$

In order for these two particles to actually scatter off each other, two conditions must be satisfied.

1. $\boldsymbol{v}_0 \cdot \boldsymbol{r}_0 < 0$ [The particles approach each other.]
2. $b = \left| \boldsymbol{r}_0 \times \hat{\boldsymbol{v}}_0 \right| < r_c$ [The particles can get within interaction range of each other.]

b is called the impact parameter and denotes the closest distance between the particles if $V(r) \equiv 0$ [see Fig.11.6a].

After the particles approach and then scatter off each other, they get eventually outside of the influence of each other so that the total energy is again purely kinetic:

$$E_{\text{tot},f} = \frac{1}{2 m_1} p_{1f}^2 + \frac{1}{2 m_2} p_{2f}^2 = \frac{1}{2M} P^2 + \frac{1}{2\mu} p_f^2$$

Since $V(r)$ is real & time-independent, H is conserved. Therefore,

$$E_{\text{tot},f} = E_{\text{tot},0}$$

Since $P = \text{const}$, E_{CM} is also conserved and so is E. Therefore,

$$\frac{1}{2\mu} p_f^2 = \frac{1}{2\mu} p_0^2$$

(11.57d)

The kinetic energy of the (effective) scattering particle is conserved, i.e., the scattering is elastic.

(11.57d) also means that the collision can only change the direction, but not the magnitude, of the relative velocity $\boldsymbol{v} = \dot{\boldsymbol{r}} = \frac{\boldsymbol{p}}{\mu}$.

The total angular momentum with respect to the origin of the lab frame is

$$\boldsymbol{L} = \boldsymbol{r}_1 \times \boldsymbol{p}_1 + \boldsymbol{r}_2 \times \boldsymbol{p}_2 \qquad L_i = \epsilon_{ijk}(r_{1j} p_{1k} + r_{2j} p_{2k})$$

$$\rightarrow \quad \dot{\boldsymbol{L}} = \{\boldsymbol{L}, H\}_{\boldsymbol{rp}} = \frac{\partial \boldsymbol{L}}{\partial r_{1j}} \frac{\partial H}{\partial p_{1j}} + \frac{\partial \boldsymbol{L}}{\partial r_{2j}} \frac{\partial H}{\partial p_{2j}} - \frac{\partial H}{\partial r_{1j}} \frac{\partial \boldsymbol{L}}{\partial p_{1j}} - \frac{\partial H}{\partial r_{2j}} \frac{\partial \boldsymbol{L}}{\partial p_{2j}} \qquad \text{(a)}$$

where $\{\ ,\ \}_{\boldsymbol{rp}}$ is the Poisson bracket.

Now,

$$\frac{\partial L_i}{\partial r_{1m}} \frac{\partial H}{\partial p_{1m}} = (\epsilon_{ijk} \delta_{jm} p_{1k}) \frac{p_{1m}}{m_1} = \frac{1}{m_1} \epsilon_{imk} p_{1k} p_{1m} = -\frac{1}{m_1} \epsilon_{ikm} p_{1k} p_{1m} = 0 \qquad \text{(b)}$$

$$\frac{\partial H}{\partial r_{1m}} \frac{\partial \boldsymbol{L}}{\partial p_{1m}} = \frac{\partial V}{\partial r_{1m}} \epsilon_{ijk} r_{1j} \delta_{km} = \epsilon_{ijm} r_{1j} \frac{\partial V}{\partial r_{1m}} = \epsilon_{ijm} r_{1j} \frac{\partial r}{\partial r_{1m}} \frac{dV}{dr} \qquad \text{(c)}$$

Using

$$\frac{\partial r}{\partial r_{1m}} = \frac{\partial \sqrt{(r_{1j} - r_{2j})(r_{1j} - r_{2j})}}{\partial r_{1m}} = \frac{1}{2r}[\delta_{jm}(r_{1j} - r_{2j}) + (r_{1j} - r_{2j})\delta_{jm}] = \frac{r_{1m} - r_{2m}}{r}$$

(c) becomes

$$\frac{\partial H}{\partial r_{1m}} \frac{\partial \boldsymbol{L}}{\partial p_{1m}} = \epsilon_{ijm} r_{1j} \frac{r_{1m} - r_{2m}}{r} \frac{dV}{dr} = -\epsilon_{ijm} r_{1j} r_{2m} \frac{1}{r} \frac{dV}{dr} \qquad \text{(d)}$$

Interchanging particle indices $1 \leftrightarrow 2$ gives

$$\frac{\partial H}{\partial r_{2m}} \frac{\partial \boldsymbol{L}}{\partial p_{2m}} = \epsilon_{ijm} r_{2j} \frac{r_{2m} - r_{1m}}{r} \frac{dV}{dr} = -\epsilon_{ijm} r_{2j} r_{1m} \frac{1}{r} \frac{dV}{dr}$$

$$= -\epsilon_{imj} r_{2m} r_{1j} \frac{1}{r} \frac{dV}{dr} \qquad [j \leftrightarrow m]$$

$$= \epsilon_{ijm} r_{2m} r_{1j} \frac{1}{r} \frac{dV}{dr} \qquad [\epsilon_{imj} = -\epsilon_{ijm}] \qquad \text{(e)}$$

Putting (b), (d) & (e) into (a) gives

$$\dot{\boldsymbol{L}} = 0$$

(11.57e)

so that the total angular momentum with respect to the origin is conserved in the lab frame.

It is straightforward to prove that (11.57e) is invariant to either a shift of coordinate origin, or a shift to another inertial frame. Therefore, the total angular momentum with respect to any point is also conserved in any inertial frame.

For the effective particle, the Hamiltonian is

$$H = \frac{1}{2\mu} p^2 + V(r) = E$$

(11.61a)

and the total angular momentum is

$$\boldsymbol{l} = \boldsymbol{r} \times \boldsymbol{p}$$

Following the same procedure for obtaining (11.57e), one can show that $\boldsymbol{l}$ is also conserved. The vectors $\boldsymbol{r}$ & $\boldsymbol{p}$ are always in the same plane perpendicular to $\boldsymbol{l}$.

Let $\boldsymbol{l} = l\,\hat{\boldsymbol{z}}$ and place the scattering in the x-y plane of the lab frame. In terms of polar coordinates defined by

$$x = r\cos\phi \qquad\qquad y = r\sin\phi$$

(11.61b)

(11.61a) becomes

$$H = \frac{1}{2\mu}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right) + V(r) = E$$

(11.61)

where

$$p_r = \mu \dot{r} \qquad\qquad p_\phi = \mu r^2 \dot{\phi}$$

(11.61c)

The Hamilton's equations are

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{\mu} \qquad\qquad \dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{\mu r^2}$$ [In agreement with (11.61c).]

$$\dot{p}_r = -\frac{\partial H}{\partial r} = -\frac{dV}{dr} \qquad\qquad \dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0$$

(11.62)

Conservation of $\boldsymbol{l}$ gives

$$p_\phi = \mu r^2 \dot{\phi} = l = \text{const}$$

(11.62a)

$$\therefore \qquad d\phi = \frac{l}{\mu r^2}\, dt = \frac{l}{r^2 p_r}\, dr$$ [(11.61c) used.]

$$= \pm \frac{l}{r^2} \frac{dr}{\sqrt{2\mu\left[E - V(r)\right] - \frac{l^2}{r^2}}}$$ [(11.61) used.]

(11.63)

where, depending on the initial conditions, the ± sign specifies either the incoming or outgoing branch of the trajectory.

Since ϕ is real, we have

$$2\mu\left[E - V(r)\right] - \frac{l^2}{r^2} \geq 0$$

The distance of closest approach $r_{\min}$ is therefore given by

$$E - V(r_{\min}) - \frac{l^2}{2\mu r_{\min}^2} = 0$$

(11.63a)

In other words, the point on the trajectory with $r = r_{\min}$ is the turning point of the effective potential

$$V_{\text{eff}}(r) = V(r) + \frac{l^2}{2\mu r^2}$$

for energy E. From (11.63), we also see that the trajectory of the particle is symmetric about the vector $\boldsymbol{r}_{\min}$.

Consider now the initial conditions as given by Fig.11.6a.

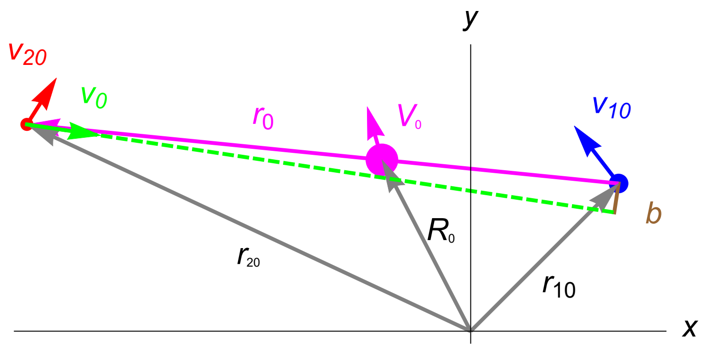

Fig.11.6a. Initial conditions in the lab frame.

The scattering of the effective particle is obtained by integrating (11.63), which, with $\phi(\infty) = \pi$, becomes

$$\phi(r) = \pi - \phi_f \pm \int_{r_{\min}}^{r} \frac{l}{r^2} \frac{dr}{\sqrt{2\mu\left[E - V(r)\right] - \frac{l^2}{r^2}}}$$

(11.63b)

where

$$\phi_f = \int_{r_{\min}}^{\infty} \frac{l}{r^2} \frac{dr}{\sqrt{2\mu\left[E - V(r)\right] - \frac{l^2}{r^2}}}$$

(11.63c)

Using (11.61b), (11.63b) can be transformed into trajectory of the effective particle.

The result for a repulsive potential $V(r) \propto \frac{1}{r^3}$ and initial conditions given by Fig.11.5 is shown in Fig.11.6(b). Using (11.56), it can be transformed into actual trajectories of particles 1 & 2 in the lab frame, as shown in Fig.11.6(c).

The scattering angle Θ of the effective particle is defined as the angle between the initial & final relative velocities [see Fig.11.6(b)]. Thus,

$\cos\Theta = \hat{\boldsymbol{v}}_f \cdot \hat{\boldsymbol{v}}_0$
(11.63e)

Later on, we shall show that Θ is equal to the CM frame scattering angle Θ_c.

In contrast, the lab frame scattering angle θ is defined as the angle between the initial & final velocities of the scattering particle, as measured in the lab frame [see Fig.11.6(c)]. In a typical scattering experiment, a beam of incident particles is trained on a target and the scattered particles are measured at various angles θ.

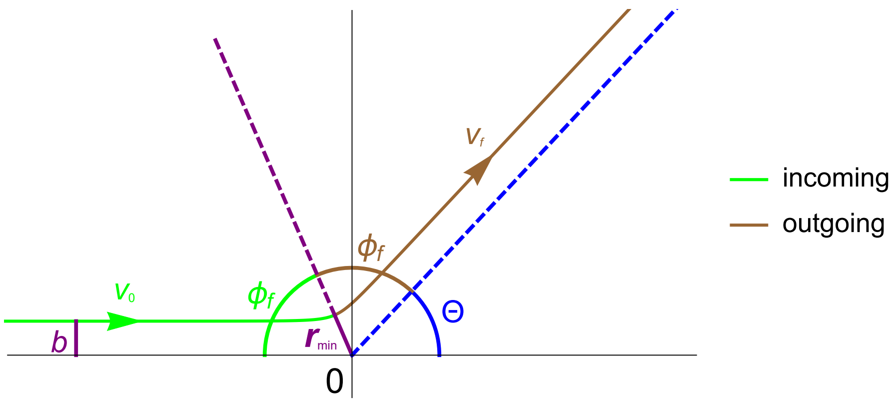

Fig.11.6 (b). Scattering of effective particle, showing the CM frame scattering angle Θ.

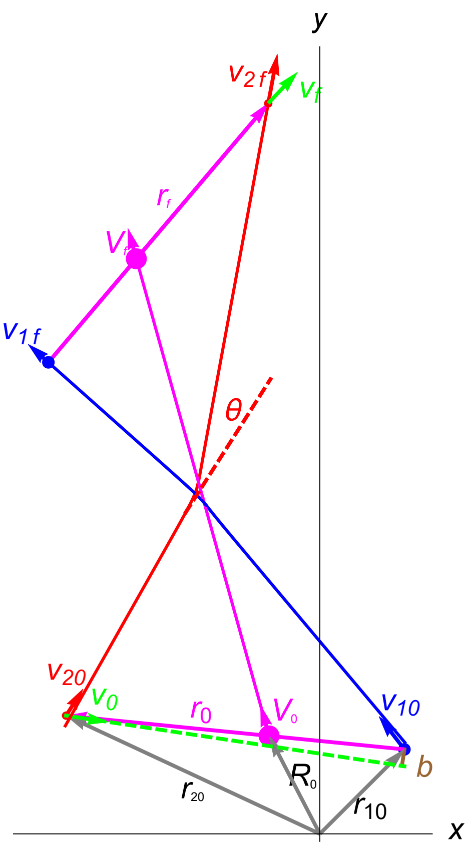

Fig.11.6(c). Scattering in the lab frame. The particles are well outside of each other's interaction range

at the initial & final configurations shown. Also shown is the lab frame scattering angle θ, assuming 2 is the scattering particle.

Consider now the scattering in the CM frame. Using the subscript c to denote quantities measured in the CM frame, we have

$$\mathbf{r}_{cj} = \mathbf{r}_j - \mathbf{R} \qquad \mathbf{v}_{cj} = \mathbf{v}_j - \mathbf{V}_{\text{CM}} \qquad j = 1, 2$$

(11.63f)

Using (11.56), we have

$$\mathbf{r}_{c1} = -\frac{m_2}{M}\mathbf{r} \qquad \mathbf{r}_{c2} = \frac{m_1}{M}\mathbf{r}$$

(11.63g)

Also,

$$\begin{aligned} \mathbf{P} &= m_1\,\mathbf{v}_1 + m_2\,\mathbf{v}_2 \\ &= m_1\,\mathbf{v}_{c1} + m_2\,\mathbf{v}_{c2} + M\,\mathbf{V}_{\text{CM}} \\ &= \mathbf{P}_c + M\,\mathbf{V}_{\text{CM}} \end{aligned}$$

where

$$\mathbf{P}_c = m_1\,\mathbf{v}_{c1} + m_2\,\mathbf{v}_{c2} = \text{total momentum in CM frame}$$

Using (11.56a), we have

$$\mathbf{P}_c = 0 \qquad \to \qquad m_1\,\mathbf{v}_{c1} = -m_2\,\mathbf{v}_{c2}$$

(11.63h)

(11.63f) gives the relative position and velocity in the CM frame as

$$\mathbf{r}_c \equiv \mathbf{r}_{c2} - \mathbf{r}_{c1} = \mathbf{r} \qquad \mathbf{v}_c \equiv \mathbf{v}_{c2} - \mathbf{v}_{c1} = \mathbf{v}$$

(11.63i)

The CM frame scattering angle Θ_c is defined as

$$\cos\Theta_c = \hat{\mathbf{v}}_{c0} \cdot \hat{\mathbf{v}}_{cf}$$

Using (11.63i), we have, as promised,

$$\Theta_c = \Theta$$

(11.63j)

Applying (11.53g) to (11.63b), we get the trajectories of the particles as seen in the CM frame [see Fig.11.6(d)].

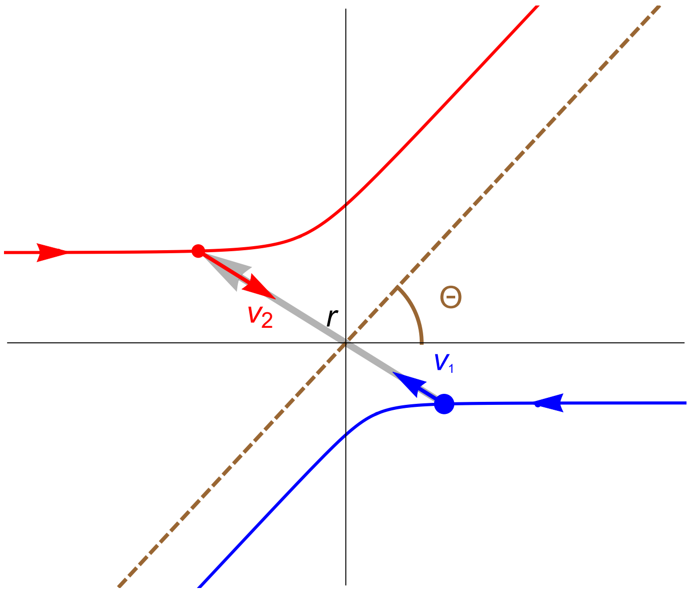

Fig.11.6(d). Scattering in the CM frame, showing the CM frame scattering angle Θ.

Finally, we wish to obtain a relation between the two scattering angles Θ & θ.

From (11.63f) & (11.56), we have

$$\mathbf{v}_2 = \mathbf{V}_{CM} + \mathbf{v}_{c2} \quad = \mathbf{V}_{CM} + \frac{m_1}{M}\mathbf{v}$$

(11.64a)

so that

$$\mathbf{v}_{20}\cdot\mathbf{v}_{2f} = V_{CM}^2 + \frac{m_1}{M}\mathbf{V}_{CM}\cdot(\mathbf{v}_0+\mathbf{v}_f) + \left(\frac{m_1}{M}\right)^2 \mathbf{v}_0\cdot\mathbf{v}_f$$

$$\rightarrow \qquad v_{20}\, v_{2f}\cos\theta = V_{CM}^2 + \frac{m_1}{M}\mathbf{V}_{CM}\cdot(\mathbf{v}_0+\mathbf{v}_f) + \left(\frac{m_1}{M}\right)^2 v_0\, v_f\cos\Theta$$

(11.64b)

Although (11.64b) can be further simplified using the conservation laws of energy and momentum, we shall consider instead the special case where the target (particle 1) is initially at rest. Setting $\mathbf{v}_{10}=0$ gives

$$\mathbf{v}_{20} = \mathbf{v}_0 \qquad\qquad \mathbf{V}_{CM} = \frac{m_2}{M}\mathbf{v}_{20}$$

$$\rightarrow \qquad \mathbf{V}_{CM}\cdot\mathbf{v}_0 = \frac{m_2}{M}v^2 \qquad\qquad \mathbf{V}_{CM}\cdot\mathbf{v}_f = \frac{m_2}{M}v_0\, v_f\cos\Theta$$

so that (11.64b) simplifies to

$$v_0\, v_{2f}\cos\theta = \left(\frac{m_2}{M}\right)^2 v_0^2 + \frac{m_1\, m_2}{M^2}v_0\,(v_0 + v_f\cos\Theta) + \left(\frac{m_1}{M}\right)^2 v_0\, v_f\cos\Theta$$

$$\rightarrow \qquad \cos\theta = \frac{1}{v_{2f}\, M}(m_2\, v_0 + m_1\, v_f\cos\Theta)$$

(11.64c)

From (11.64a), we have

$$v_{2f} = \sqrt{\mathbf{V}_{CM} + 2\frac{m_1}{M}\mathbf{V}_{CM}\cdot\mathbf{v}_f + \left(\frac{m_1}{M}\right)^2 v_f^2}$$

$$= \sqrt{\left(\frac{m_2}{M}\right)^2 v_0^2 + \left(\frac{m_1}{M}\right)^2 v_f^2 + 2\,\frac{m_1\, m_2}{M^2}\, v_0\, v_f \cos\Theta}$$

(11.64d)

Setting

$$\xi = \frac{m_2}{m_1}\,\frac{v_0}{v_f} = \frac{m_1\, m_2\, v_0}{m_1\, M\, \frac{m_1}{M}\, v_f} = \frac{\mu\, v_0}{m_1\, v_{c2f}} \qquad \boldsymbol{v}_{c2f} = \frac{m_1}{M}\,\boldsymbol{v}_f$$

(11.64e)

turns (11.64d) into

$$v_{2f} = \frac{m_1\, v_f}{M}\sqrt{1+\xi^2+2\,\xi\cos\Theta}$$

(11.64f)

(11.64c) then becomes

$$\cos\theta = \frac{\xi + \cos\Theta}{\sqrt{1+\xi^2+2\,\xi\cos\Theta}}$$

(11.66)

Note that for elastic collisions, $v_0 = v_f = v$ and (11.64e) simplifies to

$$\xi = \frac{m_2}{m_1}$$

Finally, (11.66) also applies to the general case with $\boldsymbol{v}_{10} \neq 0$ if we replace all velocities with ones relative to $\boldsymbol{v}_{10}$ [see Fig.11.7]. Since the relative velocity $\boldsymbol{v}$ is not affected by this replacement, (10.66) & (11.64e) remain the same for $\boldsymbol{v}_{10} \neq 0$.

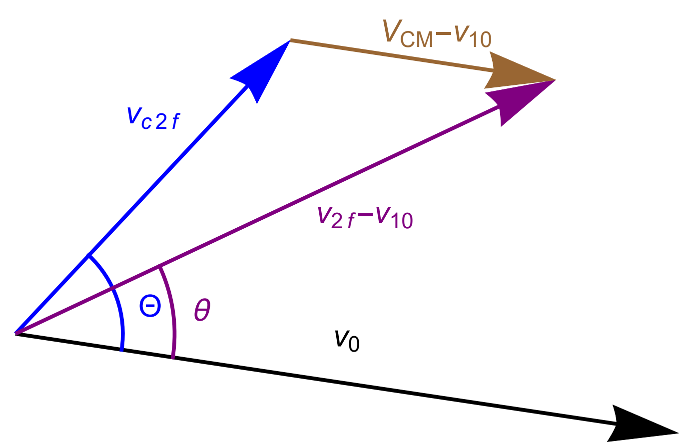

Fig.11.7. Relation between θ and Θ.

Since $\boldsymbol{V}_{CM} - \boldsymbol{v}_{10}$ is parallel to $\boldsymbol{v}_0$, we have [see Fig.11.7],

$$v_{c2f} \sin\Theta = |\, \boldsymbol{v}_{2f} - \boldsymbol{v}_{10} \,| \sin\theta$$

(11.64) $v_{c2f} \cos\Theta + V_{CM} = |\, \boldsymbol{v}_{2f} - \boldsymbol{v}_{10} \,| \cos\theta$

(11.65)

form which (11.66) can again be derived.