11.C.1.2. The Scattering Cross Section

Since the $\boldsymbol{v}_{10} = 0$ & $\boldsymbol{v}_{10} \neq 0$ results are related by a simple velocity transformation, we can, without loss of generality, restrict ourselves to the $\boldsymbol{v}_{10} = 0$ case so that $\boldsymbol{v}_{20} = \boldsymbol{v}_0$ [see Fig.11.8a].

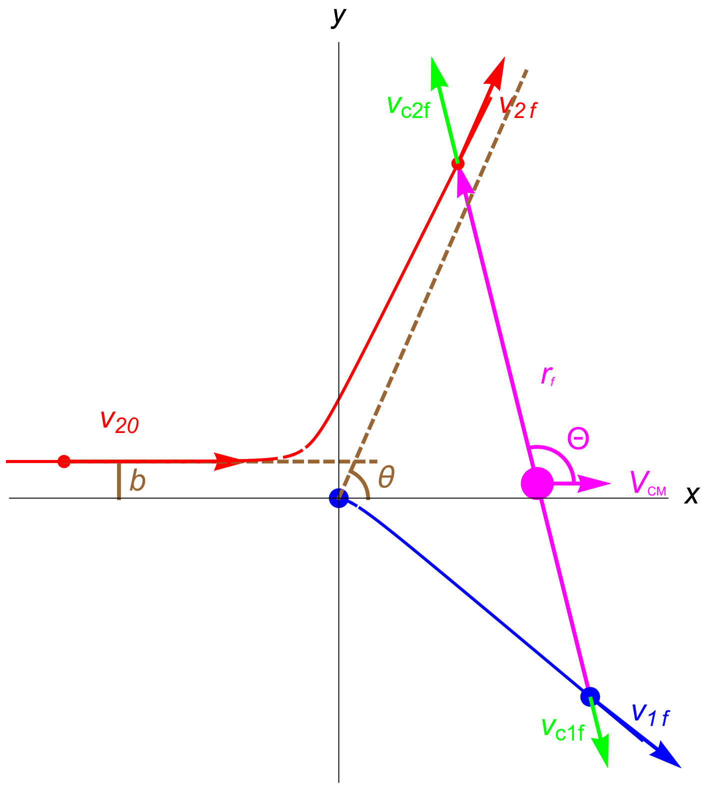

Fig.11.8a. Scattering off an initially stationary target. The curve in red (blue) is the trajectory of the scattering particle (target particle) in the lab frame. Do not confuse the red curve with that of the scattering of the effective particle.

Let

$\mathcal{I}$ = intensity of the incident beam = number of incident particles / time-area.

For a central potential, the scattering is symmetric about the x-axis, i.e., it is independent of the azimuthal angle α in the y-z plane. Consider then the particles inside a circular area element $b\, d\, b\, d\, \alpha$ of the incident beam [see Fig.11.8]. They will be scattered into the solid angle

$d\,\omega = \sin\theta\, d\,\theta\, d\,\alpha$ [lab frame]

$d\,\Omega = \sin\Theta\, d\,\Theta\, d\,\alpha$ [CM frame]

subtended at the coordinate origin of the frame. Note that α is the same in both frames.

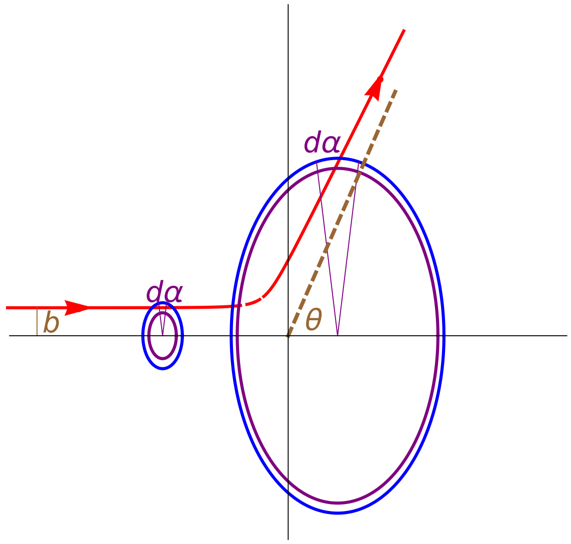

Fig.11.8. Scattering in the lab frame.

The rate $d\dot{N}$ of particles scattered into these solid angles is therefore

$$d\dot{N} = \mathcal{I}\, b\, d b\, d\alpha$$
$$= \mathcal{I} \left| \frac{b\, d b}{\sin\theta\, d\theta} \right| \sin\theta\, d\theta\, d\alpha = \mathcal{I}\, \sigma_{\text{lab}} \sin\theta\, d\theta\, d\alpha \qquad \text{[lab frame]} \qquad (11.67a)$$
$$= \mathcal{I} \left| \frac{b\, d b}{\sin\Theta\, d\Theta} \right| \sin\Theta\, d\Theta\, d\alpha = \mathcal{I}\, \sigma_{\text{CM}} \sin\Theta\, d\Theta\, d\alpha \qquad \text{[CM frame]}$$

(11.67)

where

$$\sigma_{\text{lab}} = \left| \frac{b\, d b}{\sin\theta\, d\theta} \right| = \text{differential cross-section in the lab frame} \qquad (11.68a)$$
$$\sigma_{\text{CM}} = \left| \frac{b\, d b}{\sin\Theta\, d\Theta} \right| = \text{differential cross-section in the CM frame}$$

(11.68)

By aligning the x-axis with the direction of $\boldsymbol{v}_0 = \boldsymbol{v}_{20} - \boldsymbol{v}_{10}$, these results can be applied to the general case of arbitrary particle movement that satisfies the collision criteria.

According to (11.63c), Θ depends only on l & E, or equivalently, b & $v_0 = |\, \boldsymbol{v}_{20} - \boldsymbol{v}_{10} \,|$. Therefore, we have

$$\sigma_{\text{CM}} = \sigma_{\text{CM}}(v_0, b)$$

On the other hand, θ also depends the individual orientations of the particle velocities via $\xi = \frac{V_{\text{CM}}}{v_{c2f}}$ in (11.66). Thus,

$$\sigma_{\text{lab}} = \sigma_{\text{lab}}(\boldsymbol{v}_{10}, \boldsymbol{v}_{20}, b)$$

Obviously, σ_{CM} is the preferred quantity to deal with in theoretical derivations.

The total scattering cross-section is obtained by integrating over the solid angles:

$$\sigma_{\text{tot}} = \int d\omega\, \sigma_{\text{lab}} = 2\pi \int_0^{\pi} d\theta \sin\theta\, \sigma_{\text{lab}} \qquad \text{[lab frame]} \qquad (11.69a)$$

$$= \int d\Omega\, \sigma_{CM} = 2\pi \int_0^{\pi} d\Theta \sin\Theta\, \sigma_{CM} \qquad \text{[CM frame]}$$

(11.69)

(11.68 & a) give

$$\sigma_{lab} = \left| \frac{\sin\Theta\, d\Theta}{\sin\theta\, d\theta} \right| \sigma_{CM}$$

Differentiating (11.66) gives

$$-\sin\theta\, d\theta = \frac{-\sin\Theta\, d\Theta}{\sqrt{1+\xi^2+2\xi\cos\Theta}} + \frac{(\xi+\cos\Theta)\,\xi\sin\Theta\, d\Theta}{(1+\xi^2+2\xi\cos\Theta)^{3/2}}$$

$$= -\frac{1+\xi\cos\Theta}{(1+\xi^2+2\xi\cos\Theta)^{3/2}} \sin\Theta\, d\Theta$$

$$\rightarrow \qquad \sigma_{lab} = \frac{(1+\xi^2+2\xi\cos\Theta)^{3/2}}{1+\xi\cos\Theta} \sigma_{CM}$$

(11.70)

Exercise 11.1.

Compute σ_{tot} for the elastic scattering of a point particle of mass m off an immobile hard sphere of radius R. Assume the incident particle has initial velocity $\boldsymbol{v}_0$ and impact parameter b.

Answer

Since the target is immobile, we have

$$\boldsymbol{v}_{10} = \boldsymbol{v}_{1f} = 0 \qquad\qquad m_1 = \infty$$

$$\rightarrow \qquad M = m_1 \qquad\qquad \mu = m_2$$

This lack of recoil of the target means that the scattering in the lab frame is identical to that of the scattering of the effective particle.

The interaction range for a hard sphere of radius $R_1 = R$ & point particle of radius $R_2 = 0$ is

$$r_c = R_1 + R_2 = R = r_{min}$$

where r_{min} is the minimum distance between the particles.

In an elastic collision with an immobile object, conservation of energy & momentum of the particle means

$$\boldsymbol{p}_{//f} = \boldsymbol{p}_{//0} \qquad\qquad \boldsymbol{p}_{\perp f} = -\boldsymbol{p}_{\perp 0}$$

where the subscript $//$ ($\perp$) denotes component tangential (perpendicular) to the plane of the object at contact.

With reference to Fig.a, we have

$$\phi_f = \sin^{-1}\frac{b}{R} \qquad\qquad \Theta = \pi - 2\phi_f = \theta$$

$$\rightarrow \qquad \sigma_{lab} = \sigma_{CM} \qquad\qquad \text{[see (11.68).]}$$

and

$$\sin\Theta = \sin(\pi - 2\phi_f) = \sin 2\phi_f = 2\sin\phi_f \cos\phi_f = 2\frac{b}{R}\sqrt{1-\left(\frac{b}{R}\right)^2}$$

$$\cos\Theta = \sqrt{1-\sin^2\Theta} = \sqrt{1-4\left(\frac{b}{R}\right)^2\left[1-\left(\frac{b}{R}\right)^2\right]} = 1-2\left(\frac{b}{R}\right)^2 \qquad (1a)$$

$\frac{d}{d\Theta}(1\,a)$ gives

$$-\sin\Theta = -\frac{4b}{R^2}\frac{db}{d\Theta}$$

$$\rightarrow \qquad \frac{1}{\sin\Theta}\frac{db}{d\Theta} = \frac{R^2}{4b}$$

$$\therefore \qquad \sigma_{CM} = \left|\frac{b\,db}{\sin\Theta\, d\Theta}\right| = \frac{R^2}{4} \qquad (1)$$

(11.69) then gives

$$\sigma_{tot} = 2\pi\int_0^\pi d\Theta \sin\Theta \frac{R^2}{4} = \pi R^2 \qquad (2)$$

= cross-section of target seen by the incident particle

which is why σ is called a cross-section.

Exercise 11.2.

Compute σ_{CM} for two particles of mass m_1 & m_2 interacting via a repulsive Coulomb potential $V(r) = \frac{\kappa}{r}$ with $\kappa > 0$. Assume the incident particle has initial velocity $\mathbf{v}_0$ and impact parameter b.

Answer

From (11.61) & (11.62a), we have

$$H = \frac{1}{2\mu}\left(p_r^2 + \frac{l^2}{r^2}\right) + \frac{\kappa}{r} = E \qquad l = \mu b v_0 \qquad (1)$$

$$\rightarrow \qquad p_r = \sqrt{2\mu\left(E - \frac{\kappa}{r}\right) - \frac{l^2}{r^2}} \qquad (2)$$

The minimum distance r_{min} is given by

$$2\mu\left(E - \frac{\kappa}{r_{min}}\right) - \frac{l^2}{r_{min}^2} = 0$$

$$\rightarrow \qquad 2\mu E r_{min}^2 - 2\mu\kappa r_{min} - l^2 = 0$$

$$r_{min} = \frac{1}{4\mu E}\left(2\mu\kappa + \sqrt{4\mu^2\kappa^2 + 8\mu E l^2}\right)$$

$$= \frac{\kappa}{2E}\left(1 + \sqrt{1 + \frac{2El^2}{\mu\kappa^2}}\right)$$

$$= \frac{\kappa}{2E}(1+e) \qquad (2a)$$

where

$$e = \sqrt{1 + \frac{2El^2}{\mu\kappa^2}} \qquad (2b)$$

(11.63c) becomes

$$\phi_f = \int_{r_{\min}}^{\infty} \frac{l}{r^2} \frac{dr}{\sqrt{2\mu\left(E - \frac{\kappa}{r}\right) - \frac{l^2}{r^2}}} \qquad = \frac{l}{\sqrt{2\mu E}} \int_{r_{\min}}^{\infty} \frac{dr}{r\sqrt{r^2 - \frac{\kappa}{E} r - \frac{l^2}{2\mu E}}}$$

Using

$$\int dx \frac{1}{x\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{-c}} \sin^{-1} \frac{bx+2c}{x\sqrt{b^2-4ac}} \qquad [\, b^2 - 4ac > 0 \,]$$

we have

$$\phi_f = -\sin^{-1} \left. \frac{\frac{\kappa}{E} r + \frac{l^2}{\mu E}}{r\sqrt{\left(\frac{\kappa}{E}\right)^2 + \frac{2l^2}{\mu E}}} \right|_{r_{\min}}^{\infty}$$

$$= -\sin^{-1} \left. \frac{1 + \frac{l^2}{r\kappa\mu}}{e} \right|_{r_{\min}}^{\infty} \qquad [\text{ (2b) used. }]$$

$$= -\sin^{-1}\frac{1}{e} + \sin^{-1}\frac{1 + \frac{l^2}{r_{\min}\kappa\mu}}{e}$$

$$= -\sin^{-1}\frac{1}{e} + \sin^{-1}\frac{1 + \frac{2El^2}{(1+e)\kappa^2\mu}}{e} \qquad [\text{ (2a) used. }] \qquad (2c)$$

From (2b), we have

$$\frac{2El^2}{\mu\kappa^2} = e^2 - 1 \qquad (2d)$$

so that (2c) becomes

$$\phi_f = -\sin^{-1}\frac{1}{e} + \sin^{-1}\frac{1+\frac{e^2-1}{1+e}}{e}$$
$$= -\sin^{-1}\frac{1}{e} + \frac{\pi}{2} \qquad (3)$$

From Fig.11.6(b), we have

$$\Theta = \pi - 2\,\phi_f$$

so that (3) becomes

$$-\frac{\Theta}{2} = -\sin^{-1}\frac{1}{e} \quad \rightarrow \quad \sin\frac{\Theta}{2} = \frac{1}{e} \qquad (3a)$$

$$\therefore \quad \cos\frac{\Theta}{2} = \sqrt{1-\frac{1}{e^2}}$$

$$\rightarrow \quad \cot\frac{\Theta}{2} = \sqrt{e^2-1} = \sqrt{\frac{2\,E\,l^2}{\mu\,\kappa^2}} \qquad \text{[(2d) used.]} \qquad (3b)$$

$$= \sqrt{\frac{2\,E\,\mu\,b^2\,v_0^2}{\kappa^2}} \qquad \text{[(1) used.]}$$

$$= \frac{2\,E\,b}{\kappa} \qquad [\,E = \frac{1}{2}\,\mu\,v_0^2\,]$$

$$\therefore \quad b = \frac{\kappa}{2\,E}\cot\frac{\Theta}{2} \qquad (4)$$

$$\rightarrow \quad \frac{d\,b}{d\,\Theta} = -\frac{\kappa}{4\,E}\csc^2\frac{\Theta}{2}$$

(11.68) then gives

$$\sigma_{CM} = \frac{b}{\sin\Theta}\,\frac{\kappa}{4\,E}\csc^2\frac{\Theta}{2}$$
$$= \frac{\kappa}{2\,E}\,\frac{1}{2\sin^2\frac{\Theta}{2}}\,\frac{\kappa}{4\,E}\csc^2\frac{\Theta}{2} \qquad \text{[(4) used.]}$$
$$= \left(\frac{\kappa}{2\,E}\right)^2 \frac{1}{4\sin^4\frac{\Theta}{2}} \qquad (5)$$

which is known as the Rutherford formula.