11.C.2. Derivation of the Boltzmann Equation

Mathematica code for the graphics in this section can be found in the file “ScatteringGeneralCase.nb”.

The Boltzmann equation is obtained by evaluating the term $\left.\frac{\partial f}{\partial t}\right|_{\text{coll}}$ in (11.55).

Since we are concerned only with the states of the particles when they are not interacting, we can treat them as hard spheres with cross section σ that interact only upon contact.

Consider then the particles inside the phase space volume $d\,\boldsymbol{q}\,d\,\boldsymbol{p}$ centered at the phase point $(\boldsymbol{q}, \boldsymbol{p})$. We shall assume that they are to be scattered at time t by another group of particles inside $d\,\boldsymbol{q}'\,d\,\boldsymbol{p}'$ at $(\boldsymbol{q}', \boldsymbol{p}')$.

To take advantage of the scattering results discussed earlier, we let $d\,\boldsymbol{q}$ be a sphere of radius r_s and

$$d\,\boldsymbol{q}' = 2\,\pi\,(b\,d\,b)\,(v\,d\,t) \qquad [\,\mu\,\boldsymbol{v} = \boldsymbol{p}' - \boldsymbol{p}\,] \qquad (11.71a)$$

which is the volume of a cylindrical shell of radius b, thickness $d\,b$ and height $v\,d\,t$ centered at $\boldsymbol{q}'$.

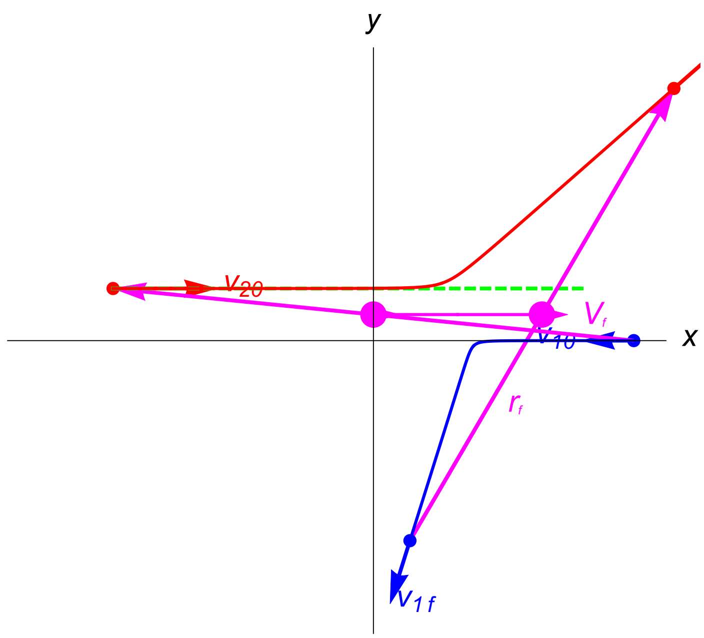

Fig.a. Scattering of 2 identical particles in lab frame with $\boldsymbol{v}_{20} = -\boldsymbol{v}_{10}$. Trajectories of the incident particle, scattered particle, and CM are shown in red, blue, and magenta, respectively.

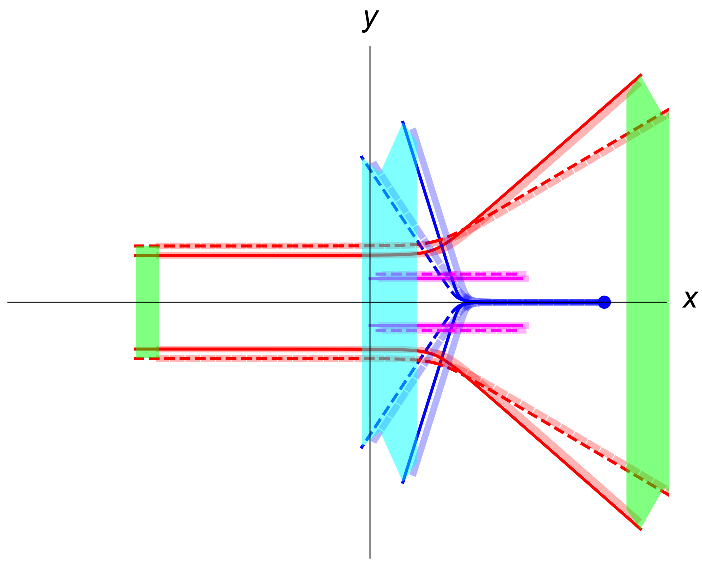

Fig.b. Scattering of 2 groups of identical particles in lab frame.
Incident particles (green strip on the left) are scattered into the green strip on the right.
Target particles (blue dot) are scattered into the cyan strip.
Each strip is the projection of a surface of revolution about the *x*-axis onto the *xy*-plane.
Vertices of the "scattered" strips indicate final positions starting from the initial positions marked by the black dots in Fig.c.
For details, see file "ScatteringGeneralCase.nb".

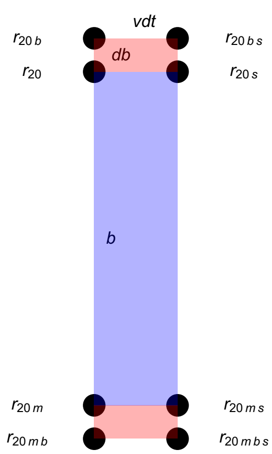

Fig.c. Projection of the incident cylindrical shell (in red) onto the xy-plane.

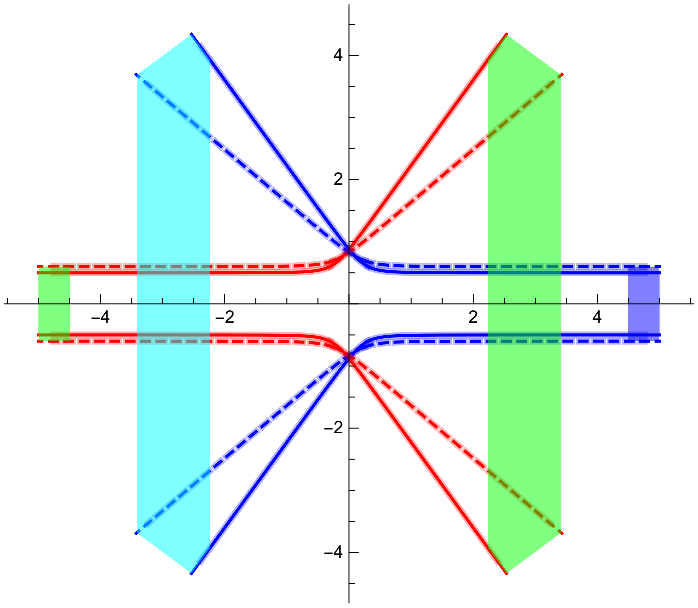

Fig.d. CM frame version of Fig.b.

Assuming all the particles in $d\boldsymbol{q}'$ are to collide with all those in $d\boldsymbol{q}$ within the time interval $(t, t+dt)$

$$|\boldsymbol{r}-\boldsymbol{r}'| \le v\,dt \quad \rightarrow \quad \boldsymbol{r} \approx \boldsymbol{r}' \quad \text{as} \quad dt \to 0$$

From (11.67), we see that the rate of particles being scattered per target particle is

$$\dot{N} = \mathcal{I} \int d\Omega\, \sigma(b, v)$$

where σ is the differential cross section into the solid angle $d\Omega$ and the intensity of the incident particles [or number of incident particles / (time · area)] is given by

$$\mathcal{I} = \rho(\boldsymbol{q}', \boldsymbol{p}', t)\, d\boldsymbol{p}' \frac{d\boldsymbol{q}'}{2\pi(b\,db)\,dt} = \rho(\boldsymbol{q}', \boldsymbol{p}', t)\, d\boldsymbol{p}'\, v \qquad \text{[(11.71a) used.]}$$

where ρ is the 1-particle probability density in phase space.

Assuming r_s to be small enough so that the same σ applies to all particles in $d\boldsymbol{q}$, the rate of target particles being scattered by all particles with momentum $\boldsymbol{p}'$ is

$$\begin{aligned}\dot{\mathcal{N}} &= \rho(\boldsymbol{q}, \boldsymbol{p}, t)\, d\boldsymbol{p}\, d\boldsymbol{q}\, \dot{N} \\ &= d\boldsymbol{p}\, d\boldsymbol{q} \int d\Omega\, \rho(\boldsymbol{q}, \boldsymbol{p}, t)\, \mathcal{I}\, \sigma(b, v) \\ &= d\boldsymbol{p}\, d\boldsymbol{q}\, d\boldsymbol{p}' \int d\Omega\, \rho(\boldsymbol{q}, \boldsymbol{p}, t)\, \rho(\boldsymbol{q}', \boldsymbol{p}', t)\, v\, \sigma(b, v) \qquad (11.71b)\end{aligned}$$

Now, the Boltzmann equation deals with transports that involve quantities in the hydrodynamic limit. In particular, the position vector $\boldsymbol{r}$ in the probability density $f(\boldsymbol{r}, \boldsymbol{p}, \boldsymbol{t})$ denotes a fluid particle that represents a spatial region $\Delta\boldsymbol{q}$ large enough to sustain local thermodynamic equilibrium. Therefore, $f(\boldsymbol{r}, \boldsymbol{p}, \boldsymbol{t})$ is defined as the average of $\rho(\boldsymbol{q}, \boldsymbol{p}, t)$ over a volume $\Delta\boldsymbol{q}$ centered at $\boldsymbol{q} = \boldsymbol{r}$.

For the collision processes in (11.71b) that affect $\left.\frac{\partial f(\boldsymbol{r}, \boldsymbol{p}, t)}{\partial t}\right|_{\text{coll}}$, we can assume

$$\boldsymbol{q}, \boldsymbol{q}' \in \Delta\boldsymbol{q}$$

so that

$$\langle \rho(\boldsymbol{q}, \boldsymbol{p}, t) \rangle_{\Delta \boldsymbol{q}} = f(\boldsymbol{r}, \boldsymbol{p}, \boldsymbol{t}) \qquad \langle \rho(\boldsymbol{q}', \boldsymbol{p}', t) \rangle_{\Delta \boldsymbol{q}} = f(\boldsymbol{r}, \boldsymbol{p}', \boldsymbol{t})$$

The rate of change in $f(\boldsymbol{r}, \boldsymbol{p}, t)$ due to particles being scattered out of $d\boldsymbol{r}$ is therefore

$$\left.\frac{\partial f(\boldsymbol{r}, \boldsymbol{p}, t)}{\partial t}\right|_{\text{out}} = -\int d\boldsymbol{p}' \int d\Omega\, f(\boldsymbol{r}, \boldsymbol{p}, t)\, f(\boldsymbol{r}, \boldsymbol{p}', t)\, v\, \sigma(b, v) \qquad (11.71c)$$

By reversing the directions of all velocities, the particles in the final states are scattered back into the respective initial states. Therefore, the rate of change in $f(\boldsymbol{r}, \boldsymbol{p}, t)$ due to particles being scattered into $d\boldsymbol{r}$ is therefore

$$\left.\frac{\partial f(\boldsymbol{r}, \boldsymbol{p}, t)}{\partial t}\right|_{\text{in}} = \int d\boldsymbol{p}' \int d\Omega\, f(\boldsymbol{r}, \boldsymbol{p}_f, t)\, f(\boldsymbol{r}, \boldsymbol{p}_f', t)\, v\, \sigma \qquad (11.71d)$$

where the subscript f denotes a quantity of the final state in the forward scattering.

Thus,

$$\left.\frac{\partial f(\boldsymbol{r}, \boldsymbol{p}, t)}{\partial t}\right|_{\text{coll}} = \int d\boldsymbol{p}' \int d\Omega \left[f(\boldsymbol{r}, \boldsymbol{p}_f, t)\, f(\boldsymbol{r}, \boldsymbol{p}_f', t) - f(\boldsymbol{r}, \boldsymbol{p}, t)\, f(\boldsymbol{r}, \boldsymbol{p}', t) \right] v\, \sigma \qquad (11.73)$$

Putting (11.73) into (11.55) gives

$$\frac{\partial f(\boldsymbol{r}, \boldsymbol{p}, t)}{\partial t} + \dot{\boldsymbol{r}} \cdot \frac{\partial f(\boldsymbol{r}, \boldsymbol{p}, t)}{\partial \boldsymbol{r}} + \dot{\boldsymbol{p}} \cdot \frac{\partial f(\boldsymbol{r}, \boldsymbol{p}, t)}{\partial \boldsymbol{p}}$$

$$= \int d\boldsymbol{p}' \int d\Omega \left[f(\boldsymbol{r}, \boldsymbol{p}_f, t)\, f(\boldsymbol{r}, \boldsymbol{p}_f', t) - f(\boldsymbol{r}, \boldsymbol{p}, t)\, f(\boldsymbol{r}, \boldsymbol{p}', t) \right] v\, \sigma \qquad (11.74)$$