11.C. The Boltzmann Equation

Read Riechl's opening statement.

Consider a dilute gas of N particles of mass m in a volume V. The inter-particle interaction potential $\phi(\,|\,\boldsymbol{q}_i - \boldsymbol{q}_j\,|\,)$ is spherically symmetric and short-ranged.

Given the N-particle probability density

$$\rho(\boldsymbol{X}^N, t) \qquad \text{where} \qquad \boldsymbol{X}^N = \{\boldsymbol{q}^N, \boldsymbol{p}^N\}$$

the 1-particle reduced probability density is defined as [see (6.53), §S6.A]

$$\rho_1(\boldsymbol{X}_1, t) = \int d\boldsymbol{X}_2 \ldots \int d\boldsymbol{X}_N \, \rho(\boldsymbol{X}^N, t)$$

(11.54a)

= probability, of finding a particle in state $\{\boldsymbol{q}, \boldsymbol{p}\}$, per unit phase space volume

From (11.54a), we can further define the 1-particle distribution function as [see (6.68), §S6.A]

$$F_1(\boldsymbol{X}, t) = V \rho_1(\boldsymbol{X}, t) \qquad \text{where} \qquad \boldsymbol{X} = \{\boldsymbol{q}, \boldsymbol{p}\}$$

= probability, of finding a particle in state $\{\boldsymbol{q}, \boldsymbol{p}\}$, per unit momentum-space volume

and the number density as

$$f(\boldsymbol{X}, t) = N \rho_1(\boldsymbol{X}, t) = \frac{N}{V} F_1(\boldsymbol{X}, t)$$

(11.54)

= number of particles in state $\boldsymbol{X}$ per unit phase space volume

Thus,

$f(\boldsymbol{X}, t)\, d\boldsymbol{X}$ = number of particles in $d\boldsymbol{X}$ centered at $\boldsymbol{X}$.

Consider now the case where the generalized coordinate $\boldsymbol{q}$ is just the position vector $\boldsymbol{r}$. Since flows in phase space is non-dissipative (or purely convective) and incompressible, we have

$$\boldsymbol{J}_f = f\,\dot{\boldsymbol{X}} \qquad \nabla_{\boldsymbol{X}} \cdot \dot{\boldsymbol{X}} = 0 \qquad \rightarrow \qquad \nabla_{\boldsymbol{X}} \cdot \boldsymbol{J}_f = \dot{\boldsymbol{X}} \cdot \nabla_{\boldsymbol{X}} f$$

so that the balance equation for $f(\boldsymbol{r}, \boldsymbol{p}, t)$ becomes

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{\boldsymbol{r}} \cdot \frac{\partial f}{\partial \boldsymbol{r}} + \dot{\boldsymbol{p}} \cdot \frac{\partial f}{\partial \boldsymbol{p}} = \left.\frac{\partial f}{\partial t}\right|_{\text{coll}}$$

(11.55)

where the source for f is written as $\left.\frac{\partial f}{\partial t}\right|_{\text{coll}}$ since it is due entirely to collisions between particles.

Note that the implicit assumption in (11.55) is that $\boldsymbol{J}_f$ represents the evolution of the system according to the ideal gas (non-interacting, collision-less) Hamiltonian, while $\left.\frac{\partial f}{\partial t}\right|_{\text{coll}}$ contains all effects of $\phi(\,|\,\boldsymbol{q}_i - \boldsymbol{q}_j\,|\,)$.