11.D.1. Kinetic Equations for a Two-Component Gas

Consider a dilute gas composed of two types of identical particles. For convenience, we shall assume that the differences between these particles are non-mechanical so that they behave the same way dynamically.

The particle types will be called normal (N) and marked (M).

Let $\sigma_{ij:kl}(b, v)$ be the CM cross section for the scattering of a pair of type i and type j particles into a pair of type k and type l particles, respectively. Since we are dealing with potential scattering between mechanically identical particles, the only non-zero cross sections are

$$\sigma_{NN:NN} = \sigma_{NM:NM} = \sigma_{MN:MN} = \sigma_{MM:MM} \equiv \sigma \tag{11.85a}$$

The Boltzmann equations are [see (11.74)]

$$\frac{\partial f_N}{\partial t} + \frac{\mathbf{p}}{m} \cdot \frac{\partial f_N}{\partial \mathbf{r}} + \dot{\mathbf{p}} \cdot \frac{\partial f_N}{\partial \mathbf{p}} = \sum_{j=N,M} \int d\mathbf{p}' \int d\Omega \left(f_{Nf}\, f_{jf}' - f_N f_j' \right) v\, \sigma_{Nj:Nj}(b, v) \tag{11.85}$$

$$\frac{\partial f_M}{\partial t} + \frac{\mathbf{p}}{m} \cdot \frac{\partial f_M}{\partial \mathbf{r}} + \dot{\mathbf{p}} \cdot \frac{\partial f_M}{\partial \mathbf{p}} = \sum_{j=N,M} \int d\mathbf{p}' \int d\Omega \left(f_{Mf}\, f_{jf}' - f_M f_j' \right) v\, \sigma_{Mj:Mj}(b, v) \tag{11.86}$$

where

$$f_j = f_j(\mathbf{r}, \mathbf{p}, t) \qquad f_j' = f_j(\mathbf{r}, \mathbf{p}', t) \quad f_{jf} = f_j(\mathbf{r}, \mathbf{p}_f, t) \quad f_{jf}' = f_j(\mathbf{r}, \mathbf{p}_f', t) \tag{11.86a}$$

In the linear response theory, we set

$$f_j(\mathbf{r}, \mathbf{p}, t) = f_j^0(\mathbf{p}) \left[1 + h_j(\mathbf{r}, \mathbf{p}, t) \right] \qquad j = N, M$$

(11.88-9)

where

$$f_j^0(\mathbf{p}) = n_{0j} \left(\frac{\beta}{2\pi m} \right)^{3/2} e^{-\beta p^2/2m} \tag{11.87a}$$

is the equilibrium 1-particle distribution of the type j particles and

$$n_{0j} = \frac{N_j}{V} \qquad N_j = \text{total number of type } j \text{ particles.}$$

From now on, we shall assume equal numbers of both types of particles so that

$$n_{0j} = \frac{N}{2V} = \frac{n_0}{2} \qquad N = \text{total number of particles.}$$

$$\rightarrow \qquad f_j^0(\mathbf{p}) = \frac{n_0}{2} \left(\frac{\beta}{2\pi m} \right)^{3/2} e^{-\beta p^2/2m} \equiv f^0(p) \tag{11.87}$$

Using the same notational conventions in (11.86a) for h_j, we have

$$f_{kf}\, f_{jf}' - f_k f_j' = f_f^0\, f_f^{0\,\prime} (1 + h_{kf})(1 + h_{jf}') - f^0 f^{0\,\prime} (1 + h_k)(1 + h_j')$$

$$= f^0 f^{0\,\prime} (h_{kf} + h_{jf}' - h_k - h_j') + O(h^2) \qquad [\; f_f^0 f_f^{0\,\prime} = f^0 f^{0\,\prime} \text{ used.}]$$

so that (11.85-6) linearize to

$$\frac{\partial h_k}{\partial t} + \frac{\mathbf{p}}{m} \cdot \frac{\partial h_k}{\partial \mathbf{r}} + \dot{\mathbf{p}} \cdot \frac{\partial h_k}{\partial \mathbf{p}} = \sum_{j=N,M} \int d\mathbf{p}' \int d\Omega\; v\, \sigma(b, v)\, f^{0\,\prime} (h_{kf} + h_{jf}' - h_k - h_j') \tag{11.90-1}$$

For $k = N$, we have

$$S_N = \sum_{j=N,M} \left(h_{Nf} + h_{jf}' - h_N - h_j' \right)$$

$$= h_{Nf} + h_{Nf}' - h_N - h_N' + h_{Nf} + h_{Mf}' - h_N - h_M'$$
$$= 2\,(h_{Nf} - h_N) + h_{Nf}' + h_{Mf}' - (h_N' + h_M')$$

For $k = M$, we have

$$S_M = \sum_{j=N,M} \left(h_{Mf} + h_{jf}' - h_M - h_j' \right)$$
$$= h_{Mf} + h_{Nf}' - h_M - h_N' + h_{Mf} + h_{Mf}' - h_M - h_M'$$
$$= 2\,(h_{Mf} - h_M) + h_{Nf}' + h_{Mf}' - (h_N' + h_M')$$

Using the same notational conventions in (11.86a) for the quantity

$$h^{\pm} = h_N \pm h_M$$

(11.92-3)

we have

$$S_N + S_M = 2\left(h_f^+ - h^+ + h_f^+{}' - h^+{}' \right)$$

$$S_N - S_M = 2\,(h_f^- - h^-)$$

(11.90-1) then give

$$\frac{\partial h^+}{\partial t} + \frac{\boldsymbol{p}}{m} \cdot \frac{\partial h^+}{\partial \boldsymbol{r}} + \dot{\boldsymbol{p}} \cdot \frac{\partial h^+}{\partial \boldsymbol{p}} = 2 \int d\,\boldsymbol{p}' \int d\,\Omega\; v\,\sigma(b,\,v)\, f^{0}{}' \left(h_f^+ - h^+ + h_f^+{}' - h^+{}' \right) \qquad (11.94)$$

$$\frac{\partial h^-}{\partial t} + \frac{\boldsymbol{p}}{m} \cdot \frac{\partial h^-}{\partial \boldsymbol{r}} + \dot{\boldsymbol{p}} \cdot \frac{\partial h^-}{\partial \boldsymbol{p}} = 2 \int d\,\boldsymbol{p}' \int d\,\Omega\; v\,\sigma(b,\,v)\, f^{0}{}' \,(h_f^- - h^-) \qquad (11.95)$$

Using the total particle distribution

$$f_T = f_N + f_M = 2\,f_0 + h_N + h_M = 2\,f_0 + h^+$$

we can write (11.94) as

$$\frac{\partial f_T}{\partial t} + \frac{\boldsymbol{p}}{m} \cdot \frac{\partial f_T}{\partial \boldsymbol{r}} + \dot{\boldsymbol{p}} \cdot \frac{\partial f_T}{\partial \boldsymbol{p}} = 2 \int d\,\boldsymbol{p}' \int d\,\Omega\; v\,\sigma(b,\,v)\; f^{0}{}' \left(h_f^+ - h^+ + h_f^+{}' - h^+{}' \right) \qquad (11.94a)$$

which is just the (linearized) Boltzmann equation for f_T.

(11.95) is called the Lorentz-Boltzmann equation and describes particle diffusion.