11.D.2. Collision Operators

(11.94) can be written as

$$\frac{\partial h^+}{\partial t} + \frac{\mathbf{p}}{m} \cdot \frac{\partial h^+}{\partial \mathbf{r}} + \dot{\mathbf{p}} \cdot \frac{\partial h^+}{\partial \mathbf{p}} = \hat{C}^+ h^+ \tag{11.96}$$

where $\hat{C}^+$ is the Boltzmann collision operator defined by [see (11.86a) for convention of notations]

$$\hat{C}^+ g(\mathbf{p}) = 2 \int d\mathbf{p}' \int d\Omega \; v\,\sigma(b,v)\, f^{0\prime} \left(g_f - g + g_f' - g' \right) \qquad \forall\, g \tag{11.97}$$

Next, we define an inner product between functions on the momentum space by

$$\langle \varphi, \chi \rangle \equiv \left(\frac{\beta}{2\pi m} \right)^{3/2} \int d\mathbf{p}\; e^{-\beta p^2/2m}\, \varphi(\mathbf{p})\, \chi(\mathbf{p}) \tag{11.98}$$

$$= \frac{2V}{N} \int d\mathbf{p}\; f^0(\mathbf{p})\, \varphi(\mathbf{p})\, \chi(\mathbf{p}) \qquad \text{[(11.87) used.]} \tag{11.98a}$$

As usual, the adjoint $\hat{O}^\dagger$ of the operator $\hat{O}$ on our inner product space is defined by

$$\left\langle \hat{O}^\dagger \varphi, \chi \right\rangle = \left\langle \varphi, \hat{O} \chi \right\rangle \tag{11.99a}$$

Putting (11.97) into (11.98) gives

$$\left\langle \varphi, \hat{C}^+ \chi \right\rangle = \frac{4V}{N} \int d\mathbf{p} \int d\mathbf{p}' \int d\Omega \; v\,\sigma(b,v)\, f^0 f^{0\prime}\, \varphi \left(\chi_f - \chi + \chi_f' - \chi' \right) \tag{11.99b}$$

and

$$\left\langle \hat{C}^+ \varphi, \chi \right\rangle = \frac{4V}{N} \int d\mathbf{p} \int d\mathbf{p}' \int d\Omega \; v\,\sigma(b,v)\, f^0 f^{0\prime} \left(\varphi_f - \varphi + \varphi_f' - \varphi' \right) \chi \tag{11.99c}$$

Now,

$$\mathcal{I} = \int d\mathbf{p} \int d\mathbf{p}' \int d\Omega \; v\,\sigma(b,v)\, f^0 f^{0\prime}$$

is invariant under the following interchange of variables:

1. $(\mathbf{p}, \mathbf{p}') \to (\mathbf{p}', \mathbf{p})$ & $(\mathbf{p}_f, \mathbf{p}_f') \to (\mathbf{p}_f', \mathbf{p}_f)$ [incident ↔ target]
2. $(\mathbf{p}, \mathbf{p}') \to (-\mathbf{p}_f, -\mathbf{p}_f')$ & $(\mathbf{p}_f, \mathbf{p}_f') \to (-\mathbf{p}, -\mathbf{p}')$ [reverse scattering]
3. $(\mathbf{p}, \mathbf{p}') \to (-\mathbf{p}_f', -\mathbf{p}_f)$ & $(\mathbf{p}_f, \mathbf{p}_f') \to (-\mathbf{p}', -\mathbf{p})$ [1+2]

Furthermore, since $\mathcal{I}$ is invariant under

$\mathbf{p} \to -\mathbf{p}$ & $\mathbf{p}' \to -\mathbf{p}'$

the reverse scatterings can be written as

2a. $(\mathbf{p}, \mathbf{p}') \to (\mathbf{p}_f, \mathbf{p}_f')$ & $(\mathbf{p}_f, \mathbf{p}_f') \to (\mathbf{p}, \mathbf{p}')$

3a. $(\mathbf{p}, \mathbf{p}') \to (\mathbf{p}_f', \mathbf{p}_f)$ & $(\mathbf{p}_f, \mathbf{p}_f') \to (\mathbf{p}', \mathbf{p})$

Therefore, (11.99b) can be written as

$$\left\langle \varphi, \hat{C}^+ \chi \right\rangle = \frac{V}{N} \int d\mathbf{p} \int d\mathbf{p}' \int d\Omega \; v\,\sigma(b,v)\, f^0 f^{0\prime}$$
$$\times \Big\{ \varphi \left(\chi_f - \chi + \chi_f' - \chi' \right) + \varphi' \left(\chi_f' - \chi' + \chi_f - \chi \right)$$
$$+ \varphi_f \left(\chi - \chi_f + \chi' - \chi_f' \right) + \varphi_f' \left(\chi' - \chi_f' + \chi - \chi_f \right) \Big\}$$

$$= -\frac{V}{N}\int d\boldsymbol{p}\int d\boldsymbol{p}'\int d\Omega\;\; v\,\sigma(b,v)\,f^0 f^{0\prime} \qquad (11.99d)$$

$$\times\left(\varphi_f-\varphi+\varphi_f'-\varphi'\right)\left(\chi_f-\chi+\chi_f'-\chi'\right)$$

$$=\left\langle \hat{C}^+\varphi,\chi\right\rangle \qquad (11.99)$$

(11.99d) gives

$$\left\langle \varphi,\hat{C}^+\varphi\right\rangle = -\frac{V}{N}\int d\boldsymbol{p}\int d\boldsymbol{p}'\int d\Omega\;\; v\,\sigma(b,v)\,f^0 f^{0\prime}\left(\varphi_f-\varphi+\varphi_f'-\varphi'\right)^2$$

$$= -\frac{1}{4}\,\frac{N}{V}\left(\frac{\beta}{2\pi m}\right)^3\int d\boldsymbol{p}\int d\boldsymbol{p}'\int d\Omega\;\; v\,\sigma(b,v)\,e^{-\beta\left(p^2+p'^2\right)/2m} \qquad (11.101)$$

$$\times\left(\varphi_f-\varphi+\varphi_f'-\varphi'\right)^2$$

where (11.87) was used.

Since every term in the integrand of (11.101) is non-negative, we have

$$\left\langle \varphi,\hat{C}^+\varphi\right\rangle \le 0 \qquad \forall\;\varphi \qquad (11.101a)$$

with

$$\left\langle \varphi^0,\hat{C}^+\varphi^0\right\rangle = 0 \qquad \Leftrightarrow \qquad \varphi_f^0-\varphi^0+\varphi_f^{0\prime}-\varphi^{0\prime}=0 \quad \forall\;\boldsymbol{p},\boldsymbol{p}' \qquad (11.101b)$$

There are 5 independent scalar solutions to (11.101b):

1. $\varphi = A = \text{const}$ $\quad\to\quad$ $\varphi_f^0-\varphi^0+\varphi_f^{0\prime}-\varphi^{0\prime} = A-A+A-A=0$
2. $\varphi = \boldsymbol{B}\cdot\boldsymbol{p}$ with $\boldsymbol{B}$ = const
   $\to\quad \varphi_f^0-\varphi^0+\varphi_f^{0\prime}-\varphi^{0\prime} = \boldsymbol{B}\cdot(\boldsymbol{p}_f-\boldsymbol{p}+\boldsymbol{p}_f'-\boldsymbol{p}')=0$ [momentum conservation]
3. $\varphi = C\,p^2$ with C = const
   $\to\quad \varphi_f^0-\varphi^0+\varphi_f^{0\prime}-\varphi^{0\prime} = C\left(p_f^2-p^2+p_f'^2-p'^2\right)=0$ [energy conservation]

The solution of (11.96) can be obtained using the eigenfunctions of $\hat{C}^+$ defined by the eigen-equation

$$\hat{C}^+\Psi=\lambda\Psi \qquad (11.100)$$

Taking the inner product gives

$$\left\langle \Psi,\hat{C}^+\Psi\right\rangle = \lambda\,\langle\Psi,\Psi\rangle$$

From (11.98), we have

$$\langle\Psi,\Psi\rangle\ge 0$$

Using (11.101a) on (11.100) then gives

$$\lambda\le 0 \qquad (11.100a)$$

which means $\hat{C}^+$ is a negative semidefinite (or nonpositive) operator. Furthermore, its $\lambda=0$ eigenvalue has a 5-fold degeneracy [see solutions of (11.101b)].

Since $\int d\boldsymbol{p}'\int d\Omega\;\; v\,\sigma(b,v)\,f^{0\prime}$ is invariant under any rotation $\hat{\mathcal{R}}$ in the momentum space, (11.97) indicates that $\hat{C}^+ g(\boldsymbol{p})$ transforms like $g(\boldsymbol{p})$ under $\hat{\mathcal{R}}$. In other word, $\hat{C}^+$ & $\hat{\mathcal{R}}$ commute and share the same eigenfunctions. Hence, the eigenfunctions Ψ of $\hat{C}^+$ take the form

$$\Psi_{\lambda l m}(\boldsymbol{p}) = R_{\lambda l}(p)\,Y_{l m}(\hat{\boldsymbol{p}}) \qquad (11.102)$$

where $Y_{lm}(\hat{\boldsymbol{p}}) = Y_{lm}(\theta,\phi)$ are the spherical harmonics (eigenfunctions of $\hat{\mathcal{R}}$).

As an example, consider now a spatially homogeneous system with no external forces. (11.96) simplifies to

$$\frac{\partial h^+}{\partial t} = \hat{C}^+ h^+$$

$$= 2 \int d\boldsymbol{p}' \int d\Omega \; v\,\sigma(b, v)\, f^{0\prime} \left(h_f^+ - h^+ + h_f^{+\prime} - h^{+\prime} \right) \qquad (11.103)$$

The solutions to (11.103) can be written as

$$h^+(t) = \sum_{\lambda, l, m} A_{\lambda l m}\, e^{\lambda t}\, \Psi_{\alpha l m}(\boldsymbol{p}) \qquad (11.104)$$

where $A_{\lambda l m}$ are constants.

The proof is simple. Taking the time derivative of (11.104) gives

$$\frac{\partial h^+}{\partial t} = \sum_{\lambda, l, m} \lambda\, A_{\lambda l m}\, e^{\lambda t}\, \Psi_{\alpha l m}(\boldsymbol{p})$$

Operating $\hat{C}^+$ on (11.104) gives

$$\hat{C}^+ h^+ = \sum_{\lambda, l, m} A_{\lambda l m}\, e^{\lambda t}\, \hat{C}^+ \Psi_{\alpha l m}(\boldsymbol{p}) \;=\; \sum_{\lambda, l, m} \lambda\, A_{\lambda l m}\, e^{\lambda t}\, \Psi_{\alpha l m}(\boldsymbol{p}) \qquad \text{[(11.100) used.]}$$

QED.

The expansion coefficients $A_{\lambda l m}$ can be obtained from the initial condition

$$h^+(0) = \sum_{\lambda, l, m} A_{\lambda l m}\, \Psi_{\alpha l m}(\boldsymbol{p})$$

Since $\lambda \le 0$,

$$h^+(\infty) = A_{0 l m}\, \Psi_{0 l m}(\boldsymbol{p})$$

Similarly, the Lorentz-Boltzmann equation (11.95) can be written as

$$\frac{\partial h^-}{\partial t} + \frac{\boldsymbol{p}}{m} \cdot \frac{\partial h^-}{\partial \boldsymbol{r}} + \dot{\boldsymbol{p}} \cdot \frac{\partial h^-}{\partial \boldsymbol{p}} = \hat{C}^- h^+ \qquad (11.105)$$

where $\hat{C}^-$ is the Lorentz-Boltzmann collision operator defined by

$$\hat{C}^- g(\boldsymbol{p}) = 2 \int d\boldsymbol{p}' \int d\Omega \; v\,\sigma(b, v)\, f^{0\prime} \left(g_f - g \right) \qquad \forall\; g \qquad (11.106)$$

The analog to (11.101a-b) is

$$\left\langle \psi, \hat{C}^- \psi \right\rangle \le 0 \qquad \forall\; \varphi \qquad (11.106a)$$

with

$$\left\langle \psi^0, \hat{C}^- \psi^0 \right\rangle = 0 \qquad \Leftrightarrow \qquad \psi_f^0 - \psi^0 = 0 \qquad \forall\; \boldsymbol{p}, \boldsymbol{p}' \qquad (11.106b)$$

The only solution to (11.106b) is

$$\psi = A = \text{const} \qquad \to \qquad \psi_f^0 - \psi^0 = A - A = 0$$

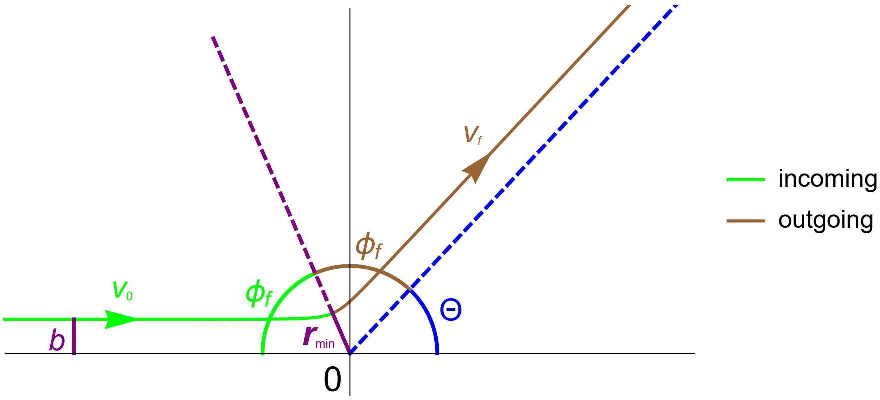
The $\lambda = 0$ eigenvalue of $\hat{C}^-$ is therefore non-degenerate.

Exercise 11.3.

Write $\left\langle \chi, \hat{C}^+ \chi \right\rangle$ in a form that makes explicit the conservation of kinetic & momentum during the scattering process.

Answer

Let the initial & final relative velocities be $\boldsymbol{v}_0 = \boldsymbol{v}_{20} - \boldsymbol{v}_{10}$ & $\boldsymbol{v}_f = \boldsymbol{v}_{2f} - \boldsymbol{v}_{1f}$, respectively. For convenience, we have reproduced Fig.11.6(b) of §11.C.1.1 below.



Let $\hat{\mathbf{e}}$ be the unit vector in the direction of $\boldsymbol{r}_{\min}$. Since $v_f = v_0$, we have

$$\boldsymbol{v}_f = \boldsymbol{v}_0 - 2\,\hat{\mathbf{e}}\left(\hat{\mathbf{e}} \cdot \boldsymbol{v}_0\right) \qquad (1a)$$

$$\boldsymbol{v}_f \cdot \boldsymbol{v}_0 = v_0^2 \cos\Theta = v_0^2 \cos(\pi - 2\,\phi_f) = -v_0^2 \cos 2\,\phi_f = v_0^2\left(1 - 2\cos^2\phi_f\right) \qquad (1)$$

Our task is to rewrite

$$\left\langle \chi, \hat{C}^+ \chi \right\rangle = -\frac{V}{N} \int d\,\boldsymbol{p} \int d\,\boldsymbol{p}' \int d\,\Omega \;\; v\,\sigma(b, v)\, f^0 f^{0\,\prime} \left(\chi_f - \chi + \chi_f' - \chi'\right)^2 \qquad (1b)$$

in a form that display explicitly the conservation of kinetic & momentum.

To begin, since $\boldsymbol{p}$ & $\boldsymbol{p}'$ in (1b) denote the initial momenta, we shall drop the subscript 0 in (1) & (1a) to make our notations consistent. Next, we rewrite $d\,\Omega \;\; v\,\sigma(b, v)$ in terms of first $\boldsymbol{P}_f$ & $\overline{\boldsymbol{p}}_f$, then $\boldsymbol{p}_f$ & $\boldsymbol{p}_f'$.

Thus,

$$v\,d\,\Omega = \frac{1}{\mu}\,d\,\Omega \int d\,\overline{p}_f\,\overline{p}_f\,\delta(\,\overline{p}_f - \overline{p}\,) \qquad \left[\; v = \frac{\overline{p}}{\mu} \text{ used.} \;\right]$$

Using

$$\delta\left(\overline{p}_f^2 - \overline{p}^{\,2}\right) = \frac{1}{|\;2\,p_f\;|}\left[\,\delta(\,\overline{p}_f - \overline{p}\,) + \delta(\,\overline{p}_f + \overline{p}\,)\,\right]$$

and that $\overline{p}_f,\ \overline{p}_f \geq 0$, we have

$$v\,d\,\Omega = \frac{2}{\mu}\,d\,\Omega \int d\,\overline{p}_f\,\overline{p}_f^2\;\delta\left(\overline{p}_f^2 - \overline{p}^{\,2}\right)$$

$$= \frac{1}{\mu^2}\,d\,\Omega \int d\,\overline{p}_f\,\overline{p}_f^2\;\delta\,(\,E_f - E) \qquad \left[\; E = \frac{\overline{p}^{\,2}}{2\,\mu} \text{ used.} \;\right]$$

$$= \frac{1}{\mu^2}\,d\,\Omega \int d\,\overline{p}_f\,\overline{p}_f^2\;\delta\,(\,E_f - E) \int d\,\boldsymbol{P}_f\,\delta(\boldsymbol{P}_f - \boldsymbol{P})$$

$$= \frac{1}{\mu^2}\,d\,\Omega \int d\,\overline{p}_f\,\overline{p}_f^2\;\delta\,(\,E_f - E) \int d\,\boldsymbol{P}_f\,\delta(\boldsymbol{P}_f - \boldsymbol{P})$$

Since $E_{\text{CM}} = \dfrac{P^2}{2M}$ is conserved, we can write

$$\begin{aligned} E_f - E &= E_f + E_{\text{CM}} - E - E_{\text{CM}} \\ &= E_{\text{tot},f} - E_{\text{tot}} \\ &= \frac{1}{2m}\left(p_f^2 + p_f'^2 - p^2 - p'^2\right) \end{aligned}$$

Also,

$$\begin{aligned} &\boldsymbol{P}_f - \boldsymbol{P} = \boldsymbol{p}_f + \boldsymbol{p}_f' - \boldsymbol{p} - \boldsymbol{p}' \\ &\int d\,\overline{\boldsymbol{p}}_f = \int d\,\Omega \int d\,\overline{p}_f\,\overline{p}_f^2 \end{aligned}$$

$$\begin{aligned} \rightarrow \quad \int v\,d\,\Omega &= \frac{2m}{\mu^2}\int d\,\overline{\boldsymbol{p}}_f\,\delta(E_f - E)\int d\,\boldsymbol{P}_f\,\delta(\boldsymbol{P}_f - \boldsymbol{P}) \\ &= \frac{8}{m}\int d\,\boldsymbol{p}_f \int d\,\boldsymbol{p}_f'\,\delta\!\left(p_f^2 + p_f'^2 - p^2 - p'^2\right)\delta(\boldsymbol{p}_f + \boldsymbol{p}_f' - \boldsymbol{p} - \boldsymbol{p}') \qquad (4) \end{aligned}$$

(1b) thus becomes

$$\begin{aligned} \left\langle \chi, \hat{C}^+ \chi \right\rangle &= -\frac{8}{m}\frac{V}{N}\int d\,\boldsymbol{p}\int d\,\boldsymbol{p}'\int d\,\boldsymbol{p}_f\int d\,\boldsymbol{p}_f'\,\delta\!\left(p_f^2 + p_f'^2 - p^2 - p'^2\right) \\ &\quad \times \delta(\boldsymbol{p}_f + \boldsymbol{p}_f' - \boldsymbol{p} - \boldsymbol{p}')\;\sigma(b, v)\,f^0 f^{0\,\prime}\left(\chi_f - \chi + \chi_f' - \chi'\right)^2 \\ &= -\frac{2}{m}\frac{N}{V}\left(\frac{\beta}{2\pi m}\right)^3\int d\,\boldsymbol{p}\int d\,\boldsymbol{p}'\int d\,\boldsymbol{p}_f\int d\,\boldsymbol{p}_f'\,\delta\!\left(p_f^2 + p_f'^2 - p^2 - p'^2\right) \qquad (6) \\ &\quad \times \delta(\boldsymbol{p}_f + \boldsymbol{p}_f' - \boldsymbol{p} - \boldsymbol{p}')\;\sigma(b, v)\,e^{-\beta\left(p^2 + p'^2\right)/2m}\left(\chi_f - \chi + \chi_f' - \chi'\right)^2 \end{aligned}$$