11.E.1. Derivation of the Diffusion Equation

The difference between the normal & marked particle densities is given by

$$m(r,t) = n_N(\boldsymbol{r},t) - n_M(\boldsymbol{r},t)$$

$$= \int d\,\boldsymbol{p}\,[\, f_N(\boldsymbol{p},\boldsymbol{r},t) - f_M(\boldsymbol{p},\boldsymbol{r},t)\,]$$

$$= \int d\,\boldsymbol{p}\, f^0(\boldsymbol{p}) \left[h_N(\boldsymbol{p},\boldsymbol{r},t) - h_M(\boldsymbol{p},\boldsymbol{r},t) \right] \qquad [\ (11.88\text{-}9)\ \text{used.}\ ]$$

$$= \int d\,\boldsymbol{p}\, f^0(\boldsymbol{p})\, h^-(\boldsymbol{p},\boldsymbol{r},t) \qquad [\ (11.93)\ \text{used.}\ ]$$

(11.107)

$$= \frac{n_0}{2} \langle 1, h^- \rangle \qquad [\ n_0 = \frac{N}{V}\ ] \qquad [\ (11.98)\ \text{used.}\ ]$$

(11.107a)

Note that

$$m(\boldsymbol{r},t) = 0 \qquad \text{at equilibrium}$$

Consider now the Lorentz-Boltzmann equation (11.95) in the absence of external forces (i.e., $\dot{\boldsymbol{p}} = 0$),

$$\frac{\partial h^-}{\partial t} + \frac{\boldsymbol{p}}{m} \cdot \frac{\partial h^-}{\partial \boldsymbol{r}} = \hat{C}^-\, h^- \qquad [\ \dot{\boldsymbol{r}} = \frac{\boldsymbol{p}}{m}\ ]$$

$$\rightarrow \qquad \frac{\partial m}{\partial t} + \int d\,\boldsymbol{p}\, f^0 \frac{\boldsymbol{p}}{m} \cdot \frac{\partial h^-}{\partial \boldsymbol{r}} = \frac{n_0}{2} \left\langle 1, \hat{C}^-\, h^- \right\rangle \qquad [\ (11.107\text{a})\ \text{used.}\ ]$$

$$= \frac{n_0}{2} \left\langle \hat{C}^-\, 1, h^- \right\rangle \qquad [\ (11.99)\ \text{used.}\ ]$$

$$= 0 \qquad [\ \hat{C}^-\, 1 = 0\ ]$$

(11.107b)

Defining the diffusion flux (diffusion current density) as

$$\boldsymbol{J}^D = \int d\,\boldsymbol{p}\, \frac{\boldsymbol{p}}{m} f^0(\boldsymbol{p})\, h^-(\boldsymbol{r},\boldsymbol{p},t) \qquad (11.109)$$

$$\rightarrow \qquad \nabla_{\boldsymbol{r}} \cdot \boldsymbol{J}^D = \int d\,\boldsymbol{p}\, f^0 \frac{\boldsymbol{p}}{m} \cdot \frac{\partial h^-}{\partial \boldsymbol{r}}$$

(11.107b) becomes

$$\frac{\partial m}{\partial t} + \nabla_{\boldsymbol{r}} \cdot \boldsymbol{J}^D = 0$$

(11.108)

According to the Fick's law [see §10.F.2]

$$\boldsymbol{J}^D = -D\, \nabla_r m$$

(11.110)

where D is the self-diffusion coefficient.

(11.108) becomes the self-diffusion equation

$$\frac{\partial m}{\partial t} = D\, \nabla_r^2 m$$

(11.111)

Using the Fourier transform

$$m(\boldsymbol{r}, t) = \int \frac{d\boldsymbol{k}}{(2\pi)^3} \int \frac{d\omega}{2\pi} e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} \overline{m}(\boldsymbol{k}, \omega) \quad (11.112)$$

on (11.111) gives

$$-i\omega\overline{m} = -D k^2 \overline{m}$$

(11.113)

$$\rightarrow \qquad \omega = -i D k^2$$

(11.114)

$$\therefore \qquad m(\boldsymbol{r}, t) \propto e^{i\boldsymbol{k}\cdot\boldsymbol{r} - D k^2 t}$$

Thus, hydrodynamic modes (long wave-lengths with $k \sim 0$) take a long time to decay. This can be attributed to the fact that particles must travel a long distance to erase a long wavelength disturbance to the equilibrium ($m = 0$) state.