11.E.2. Eigenfrequencies of the Lorentz-Boltzmann Equation

Using the Fourier transform [$(\boldsymbol{r}, t) \to (\boldsymbol{k}, \omega_n)$]

$$h^{-}(\boldsymbol{r}, \boldsymbol{p}, t) = \int \frac{d\boldsymbol{k}}{(2\pi)^3} \int \frac{d\omega_n}{2\pi} e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega_n t)} \mid \Psi_{n\boldsymbol{k}}(\boldsymbol{p}) \rangle \tag{11.116}$$

on the Lorentz-Boltzmann equation (11.105) gives

$$\left(-i\,\omega_n + i\,\boldsymbol{k}\cdot\frac{\boldsymbol{p}}{m} \right) \mid \Psi_{n\boldsymbol{k}}(\boldsymbol{p}) \rangle = \hat{C}_{\boldsymbol{p}}^{-} \mid \Psi_{n\boldsymbol{k}}(\boldsymbol{p}) \rangle \tag{11.117a}$$

where we have added a subscript $\boldsymbol{p}$ to $\hat{C}^{-}$ to emphasize the previously implicit assumption that it operates on the variable $\boldsymbol{p}$. We have also added a subscript n to ω since we shall interpret (11.117a) as an eigen-equation (for each given $\boldsymbol{k}$)

$$\left(\hat{C}_{\boldsymbol{p}}^{-} - i\,\boldsymbol{k}\cdot\frac{\boldsymbol{p}}{m} \right) \mid \Psi_{n\boldsymbol{k}}(\boldsymbol{p}) \rangle = -i\,\omega_n(\boldsymbol{k}) \mid \Psi_{n\boldsymbol{k}}(\boldsymbol{p}) \rangle \tag{11.117}$$

with eigenvalue $-i\,\omega_n(\boldsymbol{k})$ for the eigenfunction $\mid \Psi_{n\boldsymbol{k}}(\boldsymbol{p}) \rangle$. As usual, we use $n = 0, 1, 2, ...$ to label the eigenvalues in ascending order of magnitude.

For hydrodynamic modes, $k \to 0$ so that $-i\,\boldsymbol{k}\cdot\frac{\boldsymbol{p}}{m}$ can be taken as a perturbation to the unperturbed operator $\hat{C}_{\boldsymbol{p}}^{-}$.

Using k as the expansion parameter in the Rayleigh-Schrodinger perturbation theory, we set

$$\omega_n(\boldsymbol{k}) = \sum_{j=0}^{\infty} k^j\,\omega_n^{(j)} = \omega_n^{(0)} + k\,\omega_n^{(1)} + k^2\,\omega_n^{(2)} + ... \tag{11.118}$$

and

$$\mid \Psi_{n\boldsymbol{k}}(\boldsymbol{p}) \rangle = \sum_{j=0}^{\infty} k^j \left| \Psi_n^{(j)}(\boldsymbol{p}) \right\rangle = \left| \Psi_n^{(0)}(\boldsymbol{p}) \right\rangle + k \left| \Psi_n^{(1)}(\boldsymbol{p}) \right\rangle + k^2 \left| \Psi_n^{(2)}(\boldsymbol{p}) \right\rangle + ... \tag{11.119}$$

where $\omega_n^{(j)}$ & $\left| \Psi_n^{(j)}(\boldsymbol{p}) \right\rangle$ are independent of k but can depend on the direction $\hat{\boldsymbol{k}}$.

Setting $k = 0$ turns (11.117) into

$$\hat{C}_{\boldsymbol{p}}^{-} \left| \Psi_n^{(0)}(\boldsymbol{p}) \right\rangle = -i\,\omega_n^{(0)} \left| \Psi_n^{(0)}(\boldsymbol{p}) \right\rangle$$

(11.119a)

so that $\left| \Psi_n^{(0)}(\boldsymbol{p}) \right\rangle$ is the eigenfunction of $\hat{C}_{\boldsymbol{p}}^{-}$ for eigenvalue $-i\,\omega_n^{(0)}$. Since $\hat{C}_{\boldsymbol{p}}^{-}$ is Hermitian (self-adjoint) [see (11.99)], its eigenvalues must be real so that $\omega_n^{(0)}$ is purely imaginary.

To find the states $\left| \Psi_n^{(j)} \right\rangle$, we put (11.118-9) into (11.117) to get

$$\left(\hat{C}^{-} - i\,\boldsymbol{k}\cdot\frac{\boldsymbol{p}}{m} \right) \left[\left| \Psi_n^{(0)} \right\rangle + k \left| \Psi_n^{(1)} \right\rangle + k^2 \left| \Psi_n^{(2)} \right\rangle + ... \right]$$

$$= -i \left[\omega_n^{(0)} + k\,\omega_n^{(1)} + k^2\,\omega_n^{(2)} + ... \right] \left[\left| \Psi_n^{(0)} \right\rangle + k \left| \Psi_n^{(1)} \right\rangle + k^2 \left| \Psi_n^{(2)} \right\rangle + ... \right] \tag{11.119b}$$

where we have reverted to the implicit $\boldsymbol{p}$ notations.

Equating the coefficients of the same power of k on both sides of the equation gives

$$\hat{C}^{-} \left| \Psi_n^{(0)} \right\rangle = -i\,\omega_n^{(0)} \left| \Psi_n^{(0)} \right\rangle \qquad \text{[same as (11.119a)]}$$

Coefficients of k gives [$\boldsymbol{k} = k\,\hat{\boldsymbol{k}}$]

$$-i\,\hat{\boldsymbol{k}}\cdot\frac{\boldsymbol{p}}{m} \left| \Psi_n^{(0)} \right\rangle + \hat{C}^{-} \left| \Psi_n^{(1)} \right\rangle = -i \left[\omega_n^{(0)} \left| \Psi_n^{(1)} \right\rangle + \omega_n^{(1)} \left| \Psi_n^{(0)} \right\rangle \right]$$

$$\to \qquad \left[\hat{C}^{-} + i\,\omega_n^{(0)}\right] \left| \Psi_n^{(1)} \right\rangle = i\left[\hat{\boldsymbol{k}}\cdot\frac{\boldsymbol{p}}{m} - \omega_n^{(1)}\right] \left| \Psi_n^{(0)} \right\rangle$$

$$\left| \Psi_n^{(1)} \right\rangle = i\,\frac{\hat{Q}_n}{\hat{C}^{-} + i\,\omega_n^{(0)}}\left[\hat{\boldsymbol{k}}\cdot\frac{\boldsymbol{p}}{m} - \omega_n^{(1)}\right] \left| \Psi_n^{(0)} \right\rangle \qquad (11.119c)$$

where $\hat{Q}_n = 1 - \hat{P}_n$ is the exclusion operator that excludes the state $\left| \Psi_n^{(0)} \right\rangle$. Its presence is necessary since

$$\frac{1}{\hat{C}^{-} + i\,\omega_n^{(0)}} \left| \Psi_n^{(0)} \right\rangle = \frac{1}{-i\,\omega_n^{(0)} + i\,\omega_n^{(0)}} \left| \Psi_n^{(0)} \right\rangle \to \infty$$

Since $\hat{Q}_n \left| \Psi_n^{(0)} \right\rangle = 0$, (11.119c) becomes

$$\left| \Psi_n^{(1)} \right\rangle = i\,\frac{\hat{Q}_n}{\hat{C}^{-} + i\,\omega_n^{(0)}}\,\hat{\boldsymbol{k}}\cdot\frac{\boldsymbol{p}}{m} \left| \Psi_n^{(0)} \right\rangle \qquad (11.119d)$$

Owing to the presence of $\hat{Q}_n$, (11.119d) gives

$$\left\langle \Psi_n^{(0)} \middle| \Psi_n^{(1)} \right\rangle = 0$$

Following Rayleigh-Schrodinger, we assume the orthogonality conditions

$$\left\langle \Psi_n^{(i)} \middle| \Psi_n^{(j)} \right\rangle = 0 \qquad \text{if} \qquad i \neq j \qquad (11.119e)$$

$\left\langle \Psi_n^{(0)} \right|$ (11.119 b) then gives

$$\left\langle \Psi_n^{(0)} \right| \left(\hat{C}^{-} - i\,\boldsymbol{k}\cdot\frac{\boldsymbol{p}}{m}\right)\left[\left| \Psi_n^{(0)} \right\rangle + k \left| \Psi_n^{(1)} \right\rangle + k^2 \left| \Psi_n^{(2)} \right\rangle + \ldots\right]$$
$$= -i\left[\omega_n^{(0)} + k\,\omega_n^{(1)} + k^2\,\omega_n^{(2)} + \ldots\right]$$

(11.119f)

where we have assumed

$$\left\langle \Psi_n^{(0)} \middle| \Psi_n^{(0)} \right\rangle = 1$$

Equating the coefficients of the same power of k on both sides of the equation gives

$$\omega_n^{(0)} = i\left\langle \Psi_n^{(0)} \middle| \hat{C}^{-} \middle| \Psi_n^{(0)} \right\rangle$$

(11.120)

$$\omega_n^{(1)} = i\left[\left\langle \Psi_n^{(0)} \middle| \hat{C}^{-} \middle| \Psi_n^{(1)} \right\rangle - i\left\langle \Psi_n^{(0)} \middle| \hat{\boldsymbol{k}}\cdot\frac{\boldsymbol{p}}{m} \middle| \Psi_n^{(0)} \right\rangle\right] \qquad (11.121a)$$

Since $\hat{C}^{-}$ is Hermitian, we have

$$\left\langle \Psi_n^{(0)} \middle| \hat{C}^{-} \middle| \Psi_n^{(1)} \right\rangle = \left\langle \hat{C}^{-}\Psi_n^{(0)} \middle| \Psi_n^{(1)} \right\rangle$$
$$= -i\,\omega_n^{(0)} \left\langle \Psi_n^{(0)} \middle| \Psi_n^{(1)} \right\rangle \qquad \text{[(11.119a) used.]}$$
$$= 0 \qquad \text{[(11.119d) used.]}$$

Hence, (11.121a) reduces to

$$\omega_n^{(1)} = \left\langle \Psi_n^{(0)} \middle| \hat{\boldsymbol{k}}\cdot\frac{\boldsymbol{p}}{m} \middle| \Psi_n^{(0)} \right\rangle$$

(11.121)

Similarly, coefficients of k^2 gives

$$\omega_n^{(2)} = i\left[\left\langle \Psi_n^{(0)} \middle| \hat{C}^{-} \middle| \Psi_n^{(2)} \right\rangle - i\left\langle \Psi_n^{(0)} \middle| \hat{\boldsymbol{k}}\cdot\frac{\boldsymbol{p}}{m} \middle| \Psi_n^{(1)} \right\rangle\right]$$

$$= \left\langle \Psi_n^{(0)} \,\middle|\, \hat{\boldsymbol{k}}\cdot\frac{\boldsymbol{p}}{m} \,\middle|\, \Psi_n^{(1)} \right\rangle \qquad \text{[(11.119a \& d) used.]}$$

$$= \left\langle \Psi_n^{(0)} \,\middle|\, \hat{\boldsymbol{k}}\cdot\frac{\boldsymbol{p}}{m} \frac{i\,\hat{Q}_n}{\hat{C}^- + i\,\omega_n^{(0)}} \hat{\boldsymbol{k}}\cdot\frac{\boldsymbol{p}}{m} \,\middle|\, \Psi_n^{(0)} \right\rangle \tag{11.122a}$$

$$= \sum_i \sum_{j\neq n} \left\langle \Psi_n^{(0)} \,\middle|\, \hat{\boldsymbol{k}}\cdot\frac{\boldsymbol{p}}{m} \,\middle|\, \Psi_i^{(0)} \right\rangle \left\langle \Psi_i^{(0)} \,\middle|\, \frac{1}{-\omega_j^{(0)} + \omega_n^{(0)}} \,\middle|\, \Psi_j^{(0)} \right\rangle \left\langle \Psi_j^{(0)} \,\middle|\, \hat{\boldsymbol{k}}\cdot\frac{\boldsymbol{p}}{m} \,\middle|\, \Psi_n^{(0)} \right\rangle$$

$$= \sum_i \sum_{j\neq n} \left\langle \Psi_n^{(0)} \,\middle|\, \hat{\boldsymbol{k}}\cdot\frac{\boldsymbol{p}}{m} \,\middle|\, \Psi_i^{(0)} \right\rangle \frac{\delta_{ij}}{-\omega_j^{(0)} + \omega_n^{(0)}} \left\langle \Psi_j^{(0)} \,\middle|\, \hat{\boldsymbol{k}}\cdot\frac{\boldsymbol{p}}{m} \,\middle|\, \Psi_n^{(0)} \right\rangle$$

$$= \sum_{j\neq n} \left\langle \Psi_n^{(0)} \,\middle|\, \hat{\boldsymbol{k}}\cdot\frac{\boldsymbol{p}}{m} \,\middle|\, \Psi_j^{(0)} \right\rangle \frac{1}{-\omega_j^{(0)} + \omega_n^{(0)}} \left\langle \Psi_j^{(0)} \,\middle|\, \hat{\boldsymbol{k}}\cdot\frac{\boldsymbol{p}}{m} \,\middle|\, \Psi_n^{(0)} \right\rangle \tag{11.122b}$$

Another way to express (11.122a) is

$$\omega_n^{(2)} = \left\langle \Psi_n^{(0)} \,\middle|\, \left(\hat{\boldsymbol{k}}\cdot\frac{\boldsymbol{p}}{m} - \omega_n^{(1)}\right) \frac{i}{\hat{C}^- + i\,\omega_n^{(0)}} \left(\hat{\boldsymbol{k}}\cdot\frac{\boldsymbol{p}}{m} - \omega_n^{(1)}\right) \,\middle|\, \Psi_n^{(0)} \right\rangle \tag{11.122}$$

$$= \sum_j \left\langle \Psi_n^{(0)} \,\middle|\, \left(\hat{\boldsymbol{k}}\cdot\frac{\boldsymbol{p}}{m} - \omega_n^{(1)}\right) \,\middle|\, \Psi_j^{(0)} \right\rangle \frac{1}{-\omega_j^{(0)} + \omega_n^{(0)}} \left\langle \Psi_j^{(0)} \,\middle|\, \left(\hat{\boldsymbol{k}}\cdot\frac{\boldsymbol{p}}{m} - \omega_n^{(1)}\right) \,\middle|\, \Psi_n^{(0)} \right\rangle \tag{11.122c}$$

The proof is as follows. Since $\left|\, \Psi_n^{(0)} \right\rangle$ are the eigenstates of a Hermitian (or self-adjoint) operator $\hat{C}^-$ [see (11.119a)], we can assume

$$\left\langle \Psi_j^{(0)} \,\middle|\, \Psi_n^{(0)} \right\rangle = \delta_{jn}$$

Thus, for $j \neq n$,

$$\left\langle \Psi_j^{(0)} \,\middle|\, \omega_n^{(1)} \,\middle|\, \Psi_n^{(0)} \right\rangle = \omega_n^{(1)} \left\langle \Psi_j^{(0)} \,\middle|\, \Psi_n^{(0)} \right\rangle = 0 \;= \left\langle \Psi_n^{(0)} \,\middle|\, \omega_n^{(1)} \,\middle|\, \Psi_j^{(0)} \right\rangle$$

so that (11.122c) is the same as (11.122b). The $j = n$ term is effectively excluded in (11.122c) since it evaluates to

$$\left(\omega_n^{(1)} - \omega_n^{(1)}\right) \frac{1}{-\omega_n^{(0)} + \omega_n^{(0)}} \left(\omega_n^{(1)} - \omega_n^{(1)}\right) = 0$$

QED.

Consider now the $n = 0$ case. As discussed in §11.D.2,

$$\omega_0^{(0)} = 0 \qquad \text{[see (11.106a)]} \tag{11.124}$$

$$\left\langle \boldsymbol{p} \,\middle|\, \Psi_0^{(0)} \right\rangle = \Psi_0^{(0)}(\boldsymbol{p}) = A = \text{const}$$

Setting

$$\left\langle \Psi_0^{(0)} \,\middle|\, \Psi_0^{(0)} \right\rangle = 1 \qquad \to \qquad \left(\frac{\beta}{2\,\pi\,m}\right)^{3/2} \int d\,\boldsymbol{p}\; e^{-\beta p^2/2m} A^2 = 1 \qquad \to \qquad A = 1$$

(11.121) then gives

$$\omega_0^{(1)} = \left(\frac{\beta}{2\,\pi\,m}\right)^{3/2} \int d\,\boldsymbol{p}\; e^{-\beta p^2/2m}\, \hat{\boldsymbol{k}}\cdot\frac{\boldsymbol{p}}{m} = 0 \tag{11.125}$$

Similarly, (11.122) gives

$$\omega_n^{(2)} = \left(\frac{\beta}{2\,\pi\,m}\right)^{3/2} \int d\,\boldsymbol{p}\; e^{-\beta p^2/2m}\, \hat{\boldsymbol{k}}\cdot\frac{\boldsymbol{p}}{m} \frac{i}{\hat{C}^-} \hat{\boldsymbol{k}}\cdot\frac{\boldsymbol{p}}{m}$$

(11.126)

To $O(k^2)$, (11.118) thus gives

$$\omega_0(\boldsymbol{k}) = i\,k^2\,\frac{1}{m^2}\left(\frac{\beta}{2\,\pi\,m}\right)^{3/2}\int d\,\boldsymbol{p}\;e^{-\beta\,p^2/2\,m}\,\hat{\boldsymbol{k}}\cdot\boldsymbol{p}\,\frac{1}{\hat{C}^-}\,\hat{\boldsymbol{k}}\cdot\boldsymbol{p} \qquad (11.126a)$$

where $\hat{Q}_0$ is omitted since it excludes only constant terms.

Comparing with (11.114), we have

$$D \approx \frac{i}{k^2}\,\omega_0(\boldsymbol{k}) \;= -\frac{1}{m^2}\left(\frac{\beta}{2\,\pi\,m}\right)^{3/2}\int d\,\boldsymbol{p}\;e^{-\beta\,p^2/2\,m}\,\hat{\boldsymbol{k}}\cdot\boldsymbol{p}\,\frac{1}{\hat{C}^-}\,\hat{\boldsymbol{k}}\cdot\boldsymbol{p} \qquad (11.127)$$