11.F.1. Derivation of the Hydrodynamic Equations

The particle density is defined as

$$
\begin{aligned}
n(\boldsymbol{r},t) &= n_N(\boldsymbol{r},t) + n_M(\boldsymbol{r},t) \\
&= \int d\boldsymbol{p} \left[f_N(\boldsymbol{r},\boldsymbol{p},t) + f_M(\boldsymbol{r},\boldsymbol{p},t) \right] \\
&= \int d\boldsymbol{p}\, f^0(\boldsymbol{p}) \left[2 + h^+(\boldsymbol{r},\boldsymbol{p},t) \right] \qquad \text{[(11.88-9) used.]} \qquad (11.128) \\
&= \frac{n_0}{2} \langle 1,\, 2 + h^+ \rangle \qquad \left[n_0 = \frac{N}{V} \right] \\
&= n_0 + \frac{n_0}{2} \langle 1,\, h^+ \rangle
\end{aligned}
$$

Using $\hat{C}^+ 1 = 0$, the Boltzmann equation (11.96) in the absence of external forces (i.e., $\dot{\boldsymbol{p}} = 0$),

$$
\frac{\partial h^+}{\partial t} + \frac{\boldsymbol{p}}{m} \cdot \frac{\partial h^+}{\partial \boldsymbol{r}} = \hat{C}^+ h^+ \qquad (11.128a)
$$

can be written as

$$
\frac{\partial (2+h^+)}{\partial t} + \frac{\boldsymbol{p}}{m} \cdot \frac{\partial (2+h^+)}{\partial \boldsymbol{r}} = \hat{C}^+ (2+h^+) \qquad (11.128b)
$$

$\frac{n_0}{2} \langle 1,\ (11.128\,b) \rangle$ gives

$$
\begin{aligned}
\frac{\partial n}{\partial t} + \int d\boldsymbol{p}\, f^0 \frac{\boldsymbol{p}}{m} \cdot \frac{\partial h^+}{\partial \boldsymbol{r}} &= \frac{n_0}{2} \left\langle 1,\, \hat{C}^+ (2+h^+) \right\rangle \\
&= \frac{n_0}{2} \left\langle \hat{C}^+ 1,\, 2+h^+ \right\rangle = 0 \qquad (11.128c)
\end{aligned}
$$

The particle flux (current density) is defined as

$$
\begin{aligned}
\boldsymbol{J}^n(\boldsymbol{r},t) &= \int d\boldsymbol{p} \frac{\boldsymbol{p}}{m} \left[f_N(\boldsymbol{r},\boldsymbol{p},t) + f_M(\boldsymbol{r},\boldsymbol{p},t) \right] \\
&= \int d\boldsymbol{p} \frac{\boldsymbol{p}}{m} f^0(\boldsymbol{p}) \left[2 + h^+(\boldsymbol{r},\boldsymbol{p},t) \right] \\
&= \int d\boldsymbol{p} \frac{\boldsymbol{p}}{m} f^0(\boldsymbol{p})\, h^+(\boldsymbol{r},\boldsymbol{p},t) \qquad (11.130)
\end{aligned}
$$

where we have used

$$
\int d\boldsymbol{p} \frac{\boldsymbol{p}}{m} f^0(\boldsymbol{p}) = -\int d\boldsymbol{p} \frac{\boldsymbol{p}}{m} f^0(\boldsymbol{p}) \qquad [\, \boldsymbol{p} \to -\boldsymbol{p} \,]
$$

$$
= 0 \qquad [\, a = -a \ \to \ a = 0 \,]
$$

Since $\boldsymbol{r}$ & $\boldsymbol{p}$ are independent variables,

$$
\nabla_{\boldsymbol{r}} \cdot \boldsymbol{J}^n(\boldsymbol{r},t) = \int d\boldsymbol{p}\, f^0 \frac{\boldsymbol{p}}{m} \cdot \frac{\partial h^+}{\partial \boldsymbol{r}}
$$

so that (11.128c) becomes

$$
\frac{\partial n}{\partial t} + \nabla_{\boldsymbol{r}} \cdot \boldsymbol{J}^n = 0 \qquad (11.129)
$$

which can be identified as the mass balance equation [c.f. (10.3) of §10.B.1.1 with $\rho = m\,n$].

The velocity field is defined as

$$
\boldsymbol{v}(\boldsymbol{r},t) = \frac{\boldsymbol{J}^n(\boldsymbol{r},t)}{n(\boldsymbol{r},t)}
$$

which, to 1st order in h^+, becomes

$$\boldsymbol{v}(\boldsymbol{r}, t) = \frac{\boldsymbol{J}^n(\boldsymbol{r}, t)}{n_0}$$

(11.129) then simplifies to

$$\frac{\partial n}{\partial t} + n_0 \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v} = 0 \qquad (11.131)$$

which is the linearized continuity equation that describes the conservation of the total number of particles.

For the reverse scattering,

$$t \to -t \quad \Rightarrow \qquad \boldsymbol{p} \to -\boldsymbol{p}$$

(11.94) then gives

$$h^+ \to h^+$$

so that (11.130) gives

$$\boldsymbol{J}^n \to -\boldsymbol{J}^n \qquad \to \qquad \boldsymbol{v} \to -\boldsymbol{v}$$

(11.131) is therefore invariant under time reversal, i.e., it contains no irreversible effects. Irreversible phenomena, such as decay to equilibrium, must come from the other hydrodynamic equations.

Using

$$\hat{C}^+ \boldsymbol{p} = 2 \int d\boldsymbol{p}' \int d\Omega \; v\,\sigma(b, v)\, f^{0\,\prime} \left(p_f - \boldsymbol{p} + \boldsymbol{p}_f' - \boldsymbol{p}' \right)$$

$$= 0 \qquad\qquad [\text{ Momentum conservation used. }]$$

$\frac{n_0}{2m} \langle \boldsymbol{p},\ (11.128\,a) \rangle$ gives

$$\frac{\partial \boldsymbol{J}^n}{\partial t} + \int d\boldsymbol{p}\, \frac{\boldsymbol{p}}{m} f^0(\boldsymbol{p}) \frac{p_j}{m} \frac{\partial h^+}{\partial r_j} = \frac{n_0}{2m} \left\langle \boldsymbol{p}, \hat{C}^+ h^+ \right\rangle$$

$$= \frac{n_0}{2m} \left\langle \hat{C}^+ \boldsymbol{p}, h^+ \right\rangle = 0 \qquad (11.132a)$$

Defining the pressure tensor as

$$\mathbb{P}_{ij}(\boldsymbol{r}, t) = \frac{1}{m} \int d\boldsymbol{p}\, f^0(\boldsymbol{p})\, p_i\, p_j\, h^+(\boldsymbol{r}, \boldsymbol{p}, t) = \mathbb{P}_{ji} \qquad (11.133)$$

we have

$$\nabla_{\boldsymbol{r}} \cdot \mathbb{P} = \frac{1}{m} \int d\boldsymbol{p}\, f^0(\boldsymbol{p})\, p_i \frac{\partial h^+}{\partial r_i} \boldsymbol{p}$$

so that (11.132a) becomes

$$m \frac{\partial \boldsymbol{J}^n}{\partial t} + \nabla_{\boldsymbol{r}} \cdot \mathbb{P} = 0 \qquad (11.132)$$

which can be identified as the momentum balance equation in the absence of external forces. Comparing with (10.8) of §10.B.1.2, we have

$$\mathbb{P} = P\,\mathbb{I} + \rho\, \boldsymbol{v}\, \boldsymbol{v} + \boldsymbol{\Pi} \qquad [\, \rho = m\, n \,] \qquad (11.132a)$$

where the convective term $\rho\, \boldsymbol{v}\, \boldsymbol{v}$ will be neglected in the linearized equation and the stress tensor $\boldsymbol{\Pi}$ contains (irreversible) viscous effects caused by collisions.

Since the particles are free except for brief moments of collisions, the total average energy density is

$$\epsilon(\boldsymbol{r}, t) = \int d\boldsymbol{p}\, \frac{\boldsymbol{p}^2}{2m} \left[f_N(\boldsymbol{r}, \boldsymbol{p}, t) + f_M(\boldsymbol{r}, \boldsymbol{p}, t) \right]$$

$$= \int d\mathbf{p}\, \frac{\mathbf{p}^2}{2m} f^0(\mathbf{p}) \left[2 + h^+(\mathbf{r}, \mathbf{p}, t) \right]$$
$$= \frac{n_0}{2} \left\langle \frac{\mathbf{p}^2}{2m}, 2 + h^+ \right\rangle$$
$$= n_0 \left(\frac{\beta}{2\pi m} \right)^{3/2} \int d\mathbf{p}\, \frac{\mathbf{p}^2}{2m} e^{-\beta p^2/2m} + \int d\mathbf{p}\, \frac{\mathbf{p}^2}{2m} f^0(\mathbf{p})\, h^+(\mathbf{r}, \mathbf{p}, t)$$
$$= \kappa + u(\mathbf{r}, t) \qquad (11.135a)$$

where, for our ideal-gas-like system,

$$\kappa = n_0 \left(\frac{\beta}{2\pi m} \right)^{3/2} \int d\mathbf{p}\, \frac{\mathbf{p}^2}{2m} e^{-\beta p^2/2m} \qquad (11.135b)$$

can be taken as its kinetic energy density and

$$u(\mathbf{r}, t) = \int d\mathbf{p}\, \frac{\mathbf{p}^2}{2m} f^0(\mathbf{p})\, h^+(\mathbf{r}, \mathbf{p}, t) = \frac{n_0}{2} \left\langle \frac{\mathbf{p}^2}{2m}, h^+ \right\rangle \qquad (11.135)$$

its internal energy density.

Thus, $\frac{n_0}{2} \left\langle \frac{\mathbf{p}^2}{2m}, (11.128\, b) \right\rangle$ gives

$$\frac{\partial \epsilon}{\partial t} + \int d\mathbf{p}\, f^0 \frac{\mathbf{p}^2}{2m} \frac{\mathbf{p}}{m} \cdot \frac{\partial h^+}{\partial \mathbf{r}} = \frac{n_0}{2} \left\langle \frac{\mathbf{p}^2}{2m}, \hat{C}^+ (2 + h^+) \right\rangle$$
$$= \frac{n_0}{4m} \left\langle \hat{C}^+ \mathbf{p}^2, 2 + h^+ \right\rangle = 0 \qquad (11.134a)$$

where $\hat{C}^+ \mathbf{p}^2 = 0$ owing to energy conservation [see 3rd solution to (11.101b) in §11.D.2].

The energy flux (current density) is defined as

$$\mathbf{J}^\epsilon(\mathbf{r}, t) = \int d\mathbf{p}\, \frac{\mathbf{p}^2}{2m} \frac{\mathbf{p}}{m} f^0(\mathbf{p}) \left[f_N(\mathbf{r}, \mathbf{p}, t) + f_M(\mathbf{r}, \mathbf{p}, t) \right]$$
$$= \int d\mathbf{p}\, \frac{\mathbf{p}^2}{2m} \frac{\mathbf{p}}{m} f^0(\mathbf{p}) \left[2 + h^+(\mathbf{r}, \mathbf{p}, t) \right]$$
$$= \int d\mathbf{p}\, \frac{\mathbf{p}^2}{2m} \frac{\mathbf{p}}{m} f^0(\mathbf{p})\, h^+(\mathbf{r}, \mathbf{p}, t) \qquad (11.136)$$

where we have used

$$\int d\mathbf{p}\, \frac{\mathbf{p}^2}{2m} \frac{\mathbf{p}}{m} f^0(\mathbf{p}) = -\int d\mathbf{p}\, \frac{\mathbf{p}^2}{2m} \frac{\mathbf{p}}{m} f^0(\mathbf{p}) \qquad [\,\mathbf{p} \to -\mathbf{p}\,]$$
$$= 0 \qquad [\, a = -a \;\to\; a = 0 \,]$$

Since $\mathbf{r}$ & $\mathbf{p}$ are independent variables,

$$\nabla_{\mathbf{r}} \cdot \mathbf{J}^\epsilon(\mathbf{r}, t) = \int d\mathbf{p}\, f^0 \frac{\mathbf{p}^2}{2m} \frac{\mathbf{p}}{m} \cdot \frac{\partial h^+}{\partial \mathbf{r}}$$

so that (11.134a) becomes

$$\frac{\partial \epsilon}{\partial t} + \nabla_r \cdot \mathbf{J}^\epsilon = 0 \qquad (11.134)$$

which can be identified as the energy balance equation.

The average energy per particle $e(\mathbf{r}, t)$ is defined by

$$\epsilon(\mathbf{r}, t) = n(\mathbf{r}, t)\, e(\mathbf{r}, t) \qquad (11.134a)$$

$$\rightarrow \qquad \frac{\partial \epsilon}{\partial t} = \frac{\partial n}{\partial t} e + n \frac{\partial e}{\partial t}$$

$$\approx \frac{\partial n}{\partial t} e_0 + n_0 \frac{\partial e}{\partial t} \qquad \text{[Linearized.]} \qquad (11.137a)$$

where

$$e_0 = \frac{\kappa}{n_0} = \left(\frac{\beta}{2\pi m}\right)^{3/2} \int d\boldsymbol{p} \frac{\boldsymbol{p}^2}{2m} e^{-\beta p^2/2m}$$

is the average (kinetic) energy of a free particle.

(11.134) thus becomes

$$n_0 \frac{\partial e}{\partial t} + e_0 \frac{\partial n}{\partial t} + \nabla_r \cdot \boldsymbol{J}^\epsilon = 0 \qquad (11.137)$$

As in §10.B.3, we decompose the stress tensor **Π** as

$$\boldsymbol{\Pi} = -\zeta \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v} - 2\eta [\nabla_{\boldsymbol{r}} \boldsymbol{v}]^s \qquad \text{[(10.30-1) used.]}$$

$$\rightarrow \qquad \Pi_{ij} = -\delta_{ij} \zeta \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v} - \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v} \right) \qquad \text{[See Ex.10.2 of §10.B.3.]}$$

The pressure tensor (11.132a) thus becomes

$$\mathbb{P}_{ij} = P \delta_{ij} + m n v_i v_j - \delta_{ij} \zeta \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v} - \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v} \right)$$

$$\approx P \delta_{ij} - \delta_{ij} \zeta \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v} - \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v} \right) \qquad \text{[Linearized.]} \qquad (11.138)$$

$$\rightarrow \qquad \nabla_{\boldsymbol{r}} \cdot \mathbb{P} = \partial_i \mathbb{P}_{ij}$$

$$\approx \partial_j P - \zeta \partial_j \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v} - \eta \left(\frac{\partial \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v}}{\partial x_j} + \nabla_{\boldsymbol{r}}^2 v_j - \frac{2}{3} \partial_j \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v} \right)$$

$$= \nabla_{\boldsymbol{r}} P - \zeta \nabla_{\boldsymbol{r}} (\nabla_{\boldsymbol{r}} \cdot \boldsymbol{v}) - \eta \left(\nabla_{\boldsymbol{r}} (\nabla_{\boldsymbol{r}} \cdot \boldsymbol{v}) + \nabla_{\boldsymbol{r}}^2 \boldsymbol{v} - \frac{2}{3} \nabla_{\boldsymbol{r}} (\nabla_{\boldsymbol{r}} \cdot \boldsymbol{v}) \right)$$

$$= \nabla_{\boldsymbol{r}} P - \left(\zeta + \frac{1}{3} \eta \right) \nabla_{\boldsymbol{r}} (\nabla_{\boldsymbol{r}} \cdot \boldsymbol{v}) - \eta \nabla_{\boldsymbol{r}}^2 \boldsymbol{v} \qquad (11.138a)$$

Putting (11.138a) into (11.132) gives the linearized momentum balance equation

$$m n_0 \frac{\partial \boldsymbol{v}}{\partial t} = -\nabla_{\boldsymbol{r}} P + \left(\zeta + \frac{1}{3} \eta \right) \nabla_{\boldsymbol{r}} (\nabla_{\boldsymbol{r}} \cdot \boldsymbol{v}) + \eta \nabla_{\boldsymbol{r}}^2 \boldsymbol{v} \qquad (11.139)$$

Under time reversal,

$$t \rightarrow -t \qquad \boldsymbol{v}(\boldsymbol{r}, -t) = -\boldsymbol{v}(\boldsymbol{r}, t) \qquad P(\boldsymbol{r}, -t) = P(\boldsymbol{r}, t)$$

$$\rightarrow \qquad \frac{\partial \boldsymbol{v}(\boldsymbol{r}, -t)}{\partial (-t)} = \frac{\partial \boldsymbol{v}(\boldsymbol{r}, t)}{\partial t} \qquad \nabla_{\boldsymbol{r}} P(\boldsymbol{r}, -t) = \nabla_{\boldsymbol{r}} P(\boldsymbol{r}, t)$$

Hence, $\nabla_{\boldsymbol{r}} P$ is reactive (invariant under time reversal) while the rest of the R.H.S. of (11.139) is dissipative (changes sign under time reversal).

Combining (11.131) with (11.137) gives

$$n_0 \frac{\partial e}{\partial t} - n_0 e_0 \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v} + \nabla_r \cdot \boldsymbol{J}^\epsilon = 0 \qquad (11.139a)$$

On the other hand, from the 1st law

$$d U = T\, d S - P\, d V$$

we have

$$d u = T\, d s + \frac{P}{n^2}\, d n$$

where

$$u = \frac{U}{N} \qquad s = \frac{S}{N} \qquad n = \frac{N}{V}$$

Hence,

$$\frac{\partial e}{\partial t} = \frac{\partial u}{\partial t} = T \frac{\partial s}{\partial t} + \frac{P}{n^2}\frac{\partial n}{\partial t} \qquad \text{[See (11.135a).]}$$

$$\approx T_0 \frac{\partial s}{\partial t} - \frac{P_0}{n_0} \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v} \qquad \text{[(11.131) used \& linearized.]} \quad (11.139b)$$

where the subscript 0 denotes an equilibrium value.

Putting the linearized entropy balance equation [see (10.38) of §10.C.1]

$$m\, n_0 \frac{\partial s}{\partial t} = \frac{K}{T_0} \nabla_{\boldsymbol{r}}^2 T \qquad (11.143)$$

into (11.139b) gives

$$\frac{\partial e}{\partial t} \approx \frac{K}{n_0} \nabla_{\boldsymbol{r}}^2 T - \frac{P_0}{n_0} \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v} \qquad (11.143a)$$

Comparing with (11.139a) then gives

$$K \nabla_{\boldsymbol{r}}^2 T - P_0 \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v} - n_0\, e_0 \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v} + \nabla_r \cdot \boldsymbol{J}^{\epsilon} = 0$$

$$\rightarrow \qquad \boldsymbol{J}^{\epsilon} = -K \nabla_{\boldsymbol{r}} T + P_0\, \boldsymbol{v} + n_0\, e_0\, \boldsymbol{v} \qquad (11.140)$$

Note that the hydrodynamic equations (11.131, 139 & 143) are exactly the per-particle version of the linearized equations (10.36-8). We have therefore established a link between the macroscopic & microscopic descriptions of transport via the Boltzmann equation.

Choose (n, T) as the independent thermodynamic variables, we have, for the linearized equations,

$$\nabla_{\boldsymbol{r}} P = \left(\frac{\partial P}{\partial n}\right)_T^0 \nabla_{\boldsymbol{r}} n + \left(\frac{\partial P}{\partial T}\right)_n^0 \nabla_{\boldsymbol{r}} T \qquad (11.140a)$$

$$\frac{\partial s}{\partial t} = \left(\frac{\partial s}{\partial n}\right)_T^0 \frac{\partial n}{\partial t} + \left(\frac{\partial s}{\partial T}\right)_n^0 \frac{\partial T}{\partial t} \qquad (11.140b)$$

where the superscript 0 denotes equilibrium values.

The linearized hydrodynamic equations then become

$$\frac{\partial n}{\partial t} + n_0 \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v} = 0 \qquad \text{[same as (11.131)]} \qquad (11.144)$$

$$m\, n_0 \frac{\partial \boldsymbol{v}}{\partial t} = -\left(\frac{\partial P}{\partial n}\right)_T \nabla_{\boldsymbol{r}} n - \left(\frac{\partial P}{\partial T}\right)_n \nabla_{\boldsymbol{r}} T + \eta \nabla_{\boldsymbol{r}}^2 \boldsymbol{v} + \left(\zeta + \frac{1}{3}\eta\right) \nabla_{\boldsymbol{r}} (\nabla_{\boldsymbol{r}} \cdot \boldsymbol{v}) \qquad (11.145)$$

$$m\, n_0 \left(\frac{\partial s}{\partial n}\right)_T^0 \frac{\partial n}{\partial t} + m\, n_0 \left(\frac{\partial s}{\partial T}\right)_n^0 \frac{\partial T}{\partial t} = \frac{K}{T_0} \nabla_{\boldsymbol{r}}^2 T \qquad (11.146)$$

Switching to the normal mode description (or Fourier transforms), we set

$$n(\boldsymbol{r}, t) = n_{\boldsymbol{k}}(\omega)\, e^{i \boldsymbol{k} \cdot \boldsymbol{r} - i \omega t} \qquad (11.147)$$

$$\boldsymbol{v}(\boldsymbol{r},t) = \boldsymbol{v}_{\boldsymbol{k}}(\omega)\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}-i\omega t} \tag{11.148}$$
$$T(\boldsymbol{r},t) = T_{\boldsymbol{k}}(\omega)\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}-i\omega t} \tag{11.149}$$

so that (11.144-6) become

$$-i\omega\, n_{\boldsymbol{k}}(\omega) + i\, n_0\, \boldsymbol{k}\cdot\boldsymbol{v}_{\boldsymbol{k}}(\omega) = 0 \tag{11.149a}$$

$$-i\omega\, m\, n_0\, \boldsymbol{v}_{\boldsymbol{k}}(\omega) = -i\left(\frac{\partial P}{\partial n}\right)_T \boldsymbol{k}\, n_{\boldsymbol{k}}(\omega) - i\left(\frac{\partial P}{\partial T}\right)_n \boldsymbol{k}\, T_{\boldsymbol{k}}(\omega) \tag{11.149b}$$
$$-\eta k^2\, \boldsymbol{v}_{\boldsymbol{k}}(\omega) - \left(\zeta + \frac{1}{3}\eta\right)\boldsymbol{k}\,[\,\boldsymbol{k}\cdot\boldsymbol{v}_{\boldsymbol{k}}(\omega)\,]$$

$$-i\omega\, m\, n_0\left(\frac{\partial s}{\partial n}\right)_T^0 n_{\boldsymbol{k}}(\omega) - i\omega\, m\, n_0\left(\frac{\partial s}{\partial T}\right)_n^0 T_{\boldsymbol{k}}(\omega) = -\frac{K}{T_0}k^2\, T_{\boldsymbol{k}}(\omega) \tag{11.149c}$$

As in §10.C.1, we set [c.f. (10.50)]

$$\boldsymbol{v}_{\boldsymbol{k}}(\omega) = v_{\boldsymbol{k}}^{//}(\omega)\,\hat{\boldsymbol{k}} + \boldsymbol{v}_{\boldsymbol{k}}^{\perp}(\omega) \qquad [\ \hat{\boldsymbol{k}}\cdot\boldsymbol{v}_{\boldsymbol{k}}^{\perp}(\omega) = 0\ ] \tag{11.149a}$$

The transverse modes involve only (11.149b), which gives

$$-i\omega\, m\, n_0\, \boldsymbol{v}_{\boldsymbol{k}}^{\perp}(\omega) = -\eta k^2\, \boldsymbol{v}_{\boldsymbol{k}}^{\perp}(\omega)$$
$$\rightarrow \qquad \left(-i\omega + \nu_t k^2\right)\boldsymbol{v}_{\boldsymbol{k}}^{\perp}(\omega) = 0 \tag{11.150}$$

where

$$\nu_t = \frac{\eta}{m\, n_0} = \text{transverse kinetic viscosity}. \tag{11.150a}$$

(11.150) is satisfied by waves with the dispersion relation

$$\omega = -i\,\nu_t\, k^2 \tag{11.151}$$

These doubly degenerate modes are shear modes with eigenvectors lying in the plane perpendicular to the propagation direction $\hat{\boldsymbol{k}}$. Indeed, their behavior is similar to the diffusion mode $\omega_0^{(2)}$ [see (11.118) of §11.E.2]. Owing to the time dependence factor $e^{-\nu_t k^2 t}$, any short wavelength shear disturbance will damp out quickly.

The rest of the hydrodynamic equations deals only with the longitudinal modes.

$$i\omega\, n_{\boldsymbol{k}}(\omega) - i\, n_0\, k\, v_{\boldsymbol{k}}^{//}(\omega) = 0$$

$$i\omega\, m\, n_0\, v_{\boldsymbol{k}}^{//}(\omega) = i\left(\frac{\partial P}{\partial n}\right)_T k\, n_{\boldsymbol{k}}(\omega) + i\left(\frac{\partial P}{\partial T}\right)_n k\, T_{\boldsymbol{k}}(\omega)$$
$$+\eta k^2\, v_{\boldsymbol{k}}^{//}(\omega) + \left(\zeta + \frac{1}{3}\eta\right)k^2\, v_{\boldsymbol{k}}^{//}(\omega)$$

$$i\omega\, m\, n_0\left(\frac{\partial s}{\partial n}\right)_T^0 n_{\boldsymbol{k}}(\omega) + i\omega\, m\, n_0\left(\frac{\partial s}{\partial T}\right)_n^0 T_{\boldsymbol{k}}(\omega) = \frac{K}{T_0}k^2\, T_{\boldsymbol{k}}(\omega)$$

A little rearrangement gives

$$\omega\, n_{\boldsymbol{k}}(\omega) - n_0\, k\, v_{\boldsymbol{k}}^{//}(\omega) = 0$$

$$-\left(\frac{\partial P}{\partial n}\right)_T \frac{k}{m\, n_0} n_{\boldsymbol{k}}(\omega) + \left[\omega + i\left(\zeta + \frac{4}{3}\eta\right)\frac{k^2}{m\, n_0}\right] v_{\boldsymbol{k}}^{//}(\omega) - \left(\frac{\partial P}{\partial T}\right)_n \frac{k}{m\, n_0} T_{\boldsymbol{k}}(\omega) = 0$$

$$\omega\,\frac{\left(\frac{\partial s}{\partial n}\right)_T^0}{\left(\frac{\partial s}{\partial T}\right)_n^0}\, n_{\boldsymbol{k}}(\omega) + \left(\omega + i\,\frac{K}{m\, n_0\left(\frac{\partial s}{\partial T}\right)_n^0 T_0}\,k^2\right) T_{\boldsymbol{k}}(\omega) = 0$$

Using the thermodynamic relations given in §10.C.1, we have [caution: $s = \frac{S}{M} = \tilde{s}$ there & $s = \frac{S}{N} = \tilde{s}\, m$ here]

$$\frac{1}{m}\left(\frac{\partial P}{\partial n}\right)_T = \frac{c^2}{\gamma} \qquad c = \sqrt{\left(\frac{\partial P}{m\,\partial n}\right)_s^0} \qquad \gamma = \frac{c_P}{c_V}$$

$$\left(\zeta + \frac{4}{3}\eta\right)\frac{1}{m\,n_0} = \nu_l \qquad \left(\frac{\partial P}{\partial T}\right)_n = \frac{m\,n_0\,\alpha_P\,c^2}{\gamma}$$

$$\left(\frac{\partial s}{\partial n}\right)_T^0 = -\frac{m\,\alpha_P\,c^2}{n_0\,\gamma} \qquad \left(\frac{\partial s}{\partial T}\right)_n^0 = \frac{c_V}{T_0} \qquad c_P = c_V + \frac{m\,c^2\,T_0\,\alpha_P^2}{\gamma}$$

Hence, the hydrodynamic equations become

$$\omega\, n_{\boldsymbol{k}}(\omega) - n_0\, k\, v_{\boldsymbol{k}}^{/\!/}(\omega) = 0$$

$$-\frac{c^2}{\gamma}\frac{k}{n_0}\, n_{\boldsymbol{k}}(\omega) + \left(\omega + i\,\nu_l\, k^2\right) v_{\boldsymbol{k}}^{/\!/}(\omega) - \frac{\alpha_P\, c^2}{\gamma}\, k\, T_{\boldsymbol{k}}(\omega) = 0$$

$$-\omega\,\frac{\frac{m\,\alpha_P\,c^2}{n_0\,\gamma}}{c_V/T_0}\, n_{\boldsymbol{k}}(\omega) + \left(\omega + i\,\frac{K}{n_0\,c_V}\,k^2\right) T_{\boldsymbol{k}}(\omega) = 0$$

or, in matrix form,

$$\begin{pmatrix} \omega & -n_0\,k & 0 \\ -\frac{c^2}{\gamma}\frac{k}{n_0} & \omega + i\,\nu_l\,k^2 & -\frac{\alpha_P\,c^2}{\gamma}\,k \\ -\omega\,\frac{m\,\alpha_P\,c^2}{n_0\,c_V\,\gamma}\,T_0 & 0 & \omega + i\gamma\,\chi\,k^2 \end{pmatrix} \begin{pmatrix} n_{\boldsymbol{k}}(\omega) \\ v_{\boldsymbol{k}}^{/\!/}(\omega) \\ T_{\boldsymbol{k}}(\omega) \end{pmatrix} = 0 \qquad (11.152)$$

where we have used (10.62) to write

$$\chi = \frac{K}{n_0\,c_P} \qquad \gamma\,\chi = \frac{K}{n_0\,c_V} \qquad (11.152a)$$

Non-trivial solutions to (11.152) are possible only if the determinant of the matrix $\mathbb{M}(\omega)$ vanishes. The roots of

$$\det \mathbb{M}(\omega) = 0$$

thus give the normal mode dispersion relations.

Using the *Mathematica* code in §Code, we get

$$\omega_1 = -i\,\chi\,k^2 = -i\,\frac{K}{n_0\,c_P}\,k^2 \qquad (11.153)$$

[Note: we have replaced c_p with c_V to avoid confusing with c_P.]

$$\omega_\pm = \pm c\,k - i\,\frac{\chi\,c_P - c_V\,(\chi - \nu_l)}{2\,c_V}\,k^2$$

$$= \pm c\,k - i\,\frac{\nu_l + \chi\,(\gamma - 1)}{2}\,k^2$$

$$= \pm c\,k - i\,\frac{1}{2}\left[\nu_l + \left(\frac{1}{c_V} - \frac{1}{c_P}\right)\frac{K}{n_0}\right]k^2 \qquad \text{[(11.152a) used.]}$$

$$= \pm c\,k - i\,\frac{1}{2\,m\,n_0}\left[\zeta + \frac{4}{3}\eta + m\left(\frac{1}{c_V} - \frac{1}{c_P}\right)K\right]k^2 \qquad (11.154)$$

Owing to the k^2 dependence, the ω_1 mode is diffusion-like and is therefore associated with the heat propagation (wave). Similarly, the k dependence means the $\omega_\pm$ modes are sound waves. Together with the 2-fold degenerated shear wave frequency in (11.151), we have 5 normal mode frequencies for the dilute-gas-like system.

Including the diffusion frequency in (11.114), we get 6 normal mode frequencies for a 2-component system.

Code

In[1]:=
$$M = \begin{pmatrix} \omega & -n_0\,k & 0 \\ -A1\,k & \omega + i\,B1\,k^2 & -A3\,k \\ -\omega\,A2 & 0 & \omega + i\,B2\,k^2 \end{pmatrix};$$

```
sol = Solve[Det[M] == 0, ω];
```

In[3]:=
```
ωs = (ω /. sol) + O[k]^3
```

Out[3]=
$$\Bigg\{\left(\frac{\left(-(A1+A2\,A3)^3\,n_0^3\right)^{1/6}}{\sqrt{3}} + \frac{A1\,n_0 + A2\,A3\,n_0}{\sqrt{3}\,\left(-(A1+A2\,A3)^3\,n_0^3\right)^{1/6}}\right)k + \Bigg(-\frac{1}{3}\,i\,(B1+B2) + \left(i\,(A1+A2\,A3)\,(A1\,B1+A2\,A3\,B1-2\,A1\,B2+A2\,A3\,B2)\,n_0^2\right)\Big/\left(6\left(-(A1+A2\,A3)^3\,n_0^3\right)^{2/3}\right) - \left(i\,(A1\,B1\,n_0+A2\,A3\,B1\,n_0-2\,A1\,B2\,n_0+A2\,A3\,B2\,n_0)\right)\Big/\left(6\left(-(A1+A2\,A3)^3\,n_0^3\right)^{1/3}\right)\Bigg)k^2 + O[k]^3,$$

$$\left(-\frac{\left(1-i\sqrt{3}\right)\left(-(A1+A2\,A3)^3\,n_0^3\right)^{1/6}}{2\sqrt{3}} - \frac{i\left(-i+\sqrt{3}\right)(A1\,n_0+A2\,A3\,n_0)}{2\sqrt{3}\left(-(A1+A2\,A3)^3\,n_0^3\right)^{1/6}}\right)k + \Bigg(-\frac{1}{3}\,i\,(B1+B2) + \left(\left(-i+\sqrt{3}\right)(A1+A2\,A3)\,(A1\,B1+A2\,A3\,B1-2\,A1\,B2+A2\,A3\,B2)\,n_0^2\right)\Big/\left(12\left(-(A1+A2\,A3)^3\,n_0^3\right)^{2/3}\right) + \left(\left(i+\sqrt{3}\right)(A1\,B1\,n_0+A2\,A3\,B1\,n_0-2\,A1\,B2\,n_0+A2\,A3\,B2\,n_0)\right)\Big/\left(12\left(-(A1+A2\,A3)^3\,n_0^3\right)^{1/3}\right)\Bigg)k^2 + O[k]^3,$$

$$\left(-\frac{\left(1+i\sqrt{3}\right)\left(-(A1+A2\,A3)^3\,n_0^3\right)^{1/6}}{2\sqrt{3}} + \frac{i\left(i+\sqrt{3}\right)(A1\,n_0+A2\,A3\,n_0)}{2\sqrt{3}\left(-(A1+A2\,A3)^3\,n_0^3\right)^{1/6}}\right)k + \Bigg(-\frac{1}{3}\,i\,(B1+B2) - \left(\left(i+\sqrt{3}\right)(A1+A2\,A3)\,(A1\,B1+A2\,A3\,B1-2\,A1\,B2+A2\,A3\,B2)\,n_0^2\right)\Big/\left(12\left(-(A1+A2\,A3)^3\,n_0^3\right)^{2/3}\right) - \left(\left(-i+\sqrt{3}\right)(A1\,B1\,n_0+A2\,A3\,B1\,n_0-2\,A1\,B2\,n_0+A2\,A3\,B2\,n_0)\right)\Big/\left(12\left(-(A1+A2\,A3)^3\,n_0^3\right)^{1/3}\right)\Bigg)k^2 + O[k]^3\Bigg\}$$

In[4]:= `ωs // PowerExpand // FullSimplify`

Out[4]= $$\left\{\sqrt{A1+A2\,A3}\,\sqrt{n_0}\,k-\frac{i\,(A1\,B1+A2\,A3\,(B1+B2))\,k^2}{2\,(A1+A2\,A3)}+O[k]^3,\right.$$
$$\left.-\sqrt{A1+A2\,A3}\,\sqrt{n_0}\,k-\frac{i\,(A1\,B1+A2\,A3\,(B1+B2))\,k^2}{2\,(A1+A2\,A3)}+O[k]^3,\ -\frac{i\,A1\,B2\,k^2}{A1+A2\,A3}+O[k]^3\right\}$$

In[5]:= `ωse = ωs // PowerExpand // Simplify`

Out[5]= $$\left\{\sqrt{A1+A2\,A3}\,\sqrt{n_0}\,k-\frac{i\,(A1\,B1+A2\,A3\,(B1+B2))\,k^2}{2\,(A1+A2\,A3)}+O[k]^3,\right.$$
$$\left.-\sqrt{A1+A2\,A3}\,\sqrt{n_0}\,k-\frac{i\,(A1\,B1+A2\,A3\,(B1+B2))\,k^2}{2\,(A1+A2\,A3)}+O[k]^3,\ -\frac{i\,A1\,B2\,k^2}{A1+A2\,A3}+O[k]^3\right\}$$

In[6]:= $$\text{par} = \left\{A1 \to \frac{c^2}{\gamma\,n_0},\ A2 \to \frac{m\,\alpha_P\,c^2\,T_0}{n_0\,c_V\,\gamma},\ A3 \to \frac{\alpha_P\,c^2}{\gamma},\ B1 \to v_\ell,\ B2 \to \gamma\,\chi\right\};$$
`ωval = ((ωse /. par) // PowerExpand) // Simplify // Normal`

Out[7]= $$\left\{k\,\sqrt{n_0}\,\sqrt{\frac{c^2\,(\gamma\,c_V+c^2\,m\,T_0\,\alpha_P^2)}{\gamma^2\,c_V\,n_0}}-\frac{i\,k^2\,(\gamma\,c_V\,v_\ell+c^2\,m\,T_0\,(\gamma\,\chi+v_\ell)\,\alpha_P^2)}{2\,(\gamma\,c_V+c^2\,m\,T_0\,\alpha_P^2)},\right.$$
$$\left.-k\,\sqrt{n_0}\,\sqrt{\frac{c^2\,(\gamma\,c_V+c^2\,m\,T_0\,\alpha_P^2)}{\gamma^2\,c_V\,n_0}}-\frac{i\,k^2\,(\gamma\,c_V\,v_\ell+c^2\,m\,T_0\,(\gamma\,\chi+v_\ell)\,\alpha_P^2)}{2\,(\gamma\,c_V+c^2\,m\,T_0\,\alpha_P^2)},\ -\frac{i\,k^2\,\gamma^2\,\chi\,c_V}{\gamma\,c_V+c^2\,m\,T_0\,\alpha_P^2}\right\}$$

In[11]:= $$\omega v = \left(\omega val\ /.\ \alpha_P \to \sqrt{\frac{\gamma\,(c_P-c_V)}{c^2\,m\,T_0}}\right)\ /.\ \gamma \to \frac{c_P}{c_V}\ //\ \text{Simplify}\ //\ \text{PowerExpand}$$

Out[11]= $$\left\{\frac{k\,(-i\,k\,\chi\,c_P+c_V\,(2\,c+i\,k\,(\chi-v_\ell)))}{2\,c_V},\ -\frac{i\,k\,(k\,\chi\,c_P+c_V\,(-2\,i\,c+k\,(-\chi+v_\ell)))}{2\,c_V},\ -i\,k^2\,\chi\right\}$$