11.F.2. Eigenfrequencies of the Boltzmann Equation

Using the same procedure as that in §11.E.2 on the Boltzmann equation (11.96), we get

$$h^{+}(\boldsymbol{r}, \boldsymbol{p}, t) = \int \frac{d\boldsymbol{k}}{(2\pi)^3} \int \frac{d\omega_n}{2\pi} e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega_n t)} \,|\, \Phi_{n\boldsymbol{k}}(\boldsymbol{p})\rangle \tag{11.155}$$

$$\left(\hat{C}^{+}_{\boldsymbol{p}} - i\,\boldsymbol{k}\cdot\frac{\boldsymbol{p}}{m}\right) |\, \Phi_{n\boldsymbol{k}}(\boldsymbol{p})\rangle = -i\,\omega_n(\boldsymbol{k})\,|\, \Phi_{n\boldsymbol{k}}(\boldsymbol{p})\rangle \tag{11.156}$$

where $|\, \Phi_{n\boldsymbol{k}}(\boldsymbol{p})\rangle$ is a state in the Hilbert space that is the momentum space with an inner product defined by (11.98). As in §11.E.2, the $-i\,\boldsymbol{k}\cdot\frac{\boldsymbol{p}}{m}$ term will be treated as a perturbation.

Consider the (unperturbed) eigen-equation [in implicit $\boldsymbol{p}$ notations]

$$\hat{C}^{+}\,|\,\phi_n\rangle = -i\,\omega_n^0\,|\,\phi_n\rangle \tag{11.157}$$

As discussed in §11.D.2, the eigenvalue $\omega_0^0 = 0$ is 5-fold degenerated, with eigenfunctions

$$\phi_{0\alpha}(\boldsymbol{p}) = \langle \boldsymbol{p} \,|\, \phi_{0\alpha}\rangle \qquad \alpha = 1, \ldots, 5$$

being linear combinations of $\left(1,\ p_x,\ p_y,\ p_z,\ p^2\right)$ [see solutions of (11.101b)]. The coefficients of the linear combinations are determined by the orthonormality conditions

$$\langle \phi_{0\alpha} \,|\, \phi_{0\alpha'}\rangle = \left(\frac{\beta}{2\pi m}\right)^{3/2} \int d\boldsymbol{p}\; e^{-\beta p^2/2m}\, \phi_{0\alpha}(\boldsymbol{p})\, \phi_{0\alpha'}(\boldsymbol{p}) = \delta_{\alpha\alpha'} \tag{11.157a}$$

Using the *Mathematica* code

```
Assuming[ β > 0 && m > 0, Sqrt[β/(2 π m)] ∫_{-∞}^{∞} e^(-β p^2/(2 m)) # ⅆp & /@ {1, p, p^2, p^3, p^4}] // PowerExpand
```

```
{1, 0, m/β, 0, 3 m^2/β^2}
```

we have

$$\left(\frac{\beta}{2\pi m}\right)^{3/2} \int d\boldsymbol{p}\; e^{-\beta p^2/2m} = 1 \tag{11.157b}$$

$$\left(\frac{\beta}{2\pi m}\right)^{3/2} \int d\boldsymbol{p}\; p_j^n\, e^{-\beta p^2/2m} = 0 \qquad \text{if} \qquad n = \text{odd} \tag{11.157c}$$

$$\left(\frac{\beta}{2\pi m}\right)^{3/2} \int d\boldsymbol{p}\; p_j^2\, e^{-\beta p^2/2m} = \left(\frac{\beta}{2\pi m}\right)^{1/2} \int d p_j\; p_j^2\, e^{-\beta p_j^2/2m} = \frac{m}{\beta} \tag{11.157d}$$

$$\left(\frac{\beta}{2\pi m}\right)^{3/2} \int d\boldsymbol{p}\; p_i^2\, p_j^2\, e^{-\beta p^2/2m} = \left(\frac{m}{\beta}\right)^2 \qquad i \neq j$$

(11.157e)

$$\left(\frac{\beta}{2\pi m}\right)^{3/2} \int d\boldsymbol{p}\; p_j^4\, e^{-\beta p^2/2m} = \left(\frac{\beta}{2\pi m}\right)^{1/2} \int d p_j\; p_j^4\, e^{-\beta p_j^2/2m} = 3\left(\frac{m}{\beta}\right)^2 \tag{11.157f}$$

It is then trivial to verify that

$$\phi_{01}(\boldsymbol{p}) = 1$$

(11.158)

$$\phi_{02}(\boldsymbol{p}) = \sqrt{\frac{\beta}{m}}\ p_x$$

(11.159)

$$\phi_{03}(\boldsymbol{p}) = \sqrt{\frac{\beta}{m}}\ p_y$$

(11.160)

$$\phi_{04}(\boldsymbol{p}) = \sqrt{\frac{\beta}{m}}\ p_z$$

(11.161)

satisfy the orthonormality conditions (11.157a).

$\phi_{05}(\boldsymbol{p})$ cannot be simply proportional to p^2 since it is then not orthogonal to $\phi_{01}(\boldsymbol{p})$. Therefore, we set

$$\phi_{05}(\boldsymbol{p}) = a + b\,p^2 = a + b\left(p_x^2 + p_y^2 + p_z^2\right)$$

so that

$$[\,\phi_{05}(\boldsymbol{p})\,]^2 = a^2 + 2\,a\,b\,p^2 + b^2\,p^4$$
$$= a^2 + 2\,a\,b\left(p_x^2 + p_y^2 + p_z^2\right) + b^2\left(p_x^2 + p_y^2 + p_z^2\right)^2$$
$$= a^2 + 2\,a\,b\left(p_x^2 + p_y^2 + p_z^2\right) + b^2\left(p_x^4 + p_y^4 + p_z^4 + 2\,p_x^2\,p_y^2 + 2\,p_y^2\,p_z^2 + 2\,p_z^2\,p_x^2\right)$$

$$\rightarrow \qquad \langle\,\phi_{05}\,|\,\phi_{05}\,\rangle = a^2 + 6\,a\,b\,\frac{m}{\beta} + b^2\left[9\left(\frac{m}{\beta}\right)^2 + 6\left(\frac{m}{\beta}\right)^2\right] = 1$$

$$\langle\,\phi_{01}\,|\,\phi_{05}\,\rangle = a + 3\,b\,\frac{m}{\beta} = 0$$

$$\langle\,\phi_{0\,\alpha}\,|\,\phi_{05}\,\rangle = 0 \qquad \text{for} \qquad \alpha = 2, 3, 4$$

Using the following *Mathematica* code to solve for (a, b),

```
Solve[{a^2 + 6 a b m/β + 15 b^2 (m/β)^2 == 1, a + 3 b m/β == 0}, {a, b}]
```

$$\left\{\left\{a \to -\sqrt{\frac{3}{2}},\ b \to \frac{\beta}{\sqrt{6}\ m}\right\},\ \left\{a \to \sqrt{\frac{3}{2}},\ b \to -\frac{\beta}{\sqrt{6}\ m}\right\}\right\}$$

we have

$$\phi_{05}(\boldsymbol{p}) = -\sqrt{\frac{3}{2}} + \frac{\beta}{\sqrt{6}\ m}\,p^2$$
$$= \sqrt{\frac{2}{3}}\left(-\frac{3}{2} + \frac{\beta}{2\,m}\,p^2\right) \qquad (11.162)$$

Following §11.E.2, we set [see (11.118-9)]

$$\omega_n(\boldsymbol{k}) = \sum_{j=0}^{\infty} k^j\,\omega_n^{(j)} = \omega_n^{(0)} + k\,\omega_n^{(1)} + k^2\,\omega_n^{(2)} + \ldots \qquad (11.163a)$$

and

$$| \Phi_{n\boldsymbol{k}} \rangle = \sum_{j=0}^{\infty} k^j \, | \Phi_n^{(j)} \rangle = | \Phi_n^{(0)} \rangle + k \, | \Phi_n^{(1)} \rangle + k^2 \, | \Phi_n^{2)} \rangle + \ldots \tag{11.163b}$$

For $n = 0$, the 5-fold degeneracy of the unperturbed state requires a more elaborate treatment:

$$| \Phi_{0\alpha\boldsymbol{k}} \rangle = | \Phi_{0\alpha}^{(0)} \rangle + k \, | \Phi_{0\alpha}^{(1)} \rangle + k^2 \, | \Phi_{0\alpha}^{2)} \rangle + \ldots \qquad \alpha = 1, \ \ldots, 5$$

(11.163)

where

$$| \Phi_{0\alpha}^{(0)} \rangle = \sum_{\alpha'=1}^{5} c_{\alpha\alpha'} \, | \phi_{0\alpha'} \rangle$$

(11.164)

Furthermore, owing to the possible lifting of the degeneracy by the perturbation, (11.163a) is amended to read

$$\omega_{0\alpha}(\boldsymbol{k}) = k\,\omega_{0\alpha}^{(1)} + k^2\,\omega_{0\alpha}^{(2)} + \ldots \qquad [\ \omega_{0\alpha}^{(0)} = \omega_0^{(0)} = 0 \quad \forall\ \alpha\ ] \tag{11.165}$$

To find the states $| \Phi_{0\alpha}^{(j)} \rangle$, we put (11.163-5) into (11.156) to get

$$\left(\hat{C}^+ - i\,\boldsymbol{k} \cdot \frac{\boldsymbol{p}}{m} \right) \left[\ | \Phi_{0\alpha}^{(0)} \rangle + k \, | \Phi_{0\alpha}^{(1)} \rangle + k^2 \, | \Phi_{0\alpha}^{2)} \rangle + \ldots \right]$$
$$= -i \left[k\,\omega_{0\alpha}^{(1)} + k^2\,\omega_{0\alpha}^{(2)} + \ldots \right] \left[\ | \Phi_{0\alpha}^{(0)} \rangle + k \, | \Phi_{0\alpha}^{(1)} \rangle + k^2 \, | \Phi_{0\alpha}^{2)} \rangle + \ldots \right] \tag{11.166a}$$

Equating the coefficients of the same power of k on both sides of the equation gives

$$\hat{C}^+ \, | \Phi_{0\alpha}^{(0)} \rangle = 0 \qquad [\ n = 0 \text{ case of (11.157)}\ ]$$

(11.166b)

Coefficients of k gives [$\boldsymbol{k} = k\,\hat{\boldsymbol{k}}$],

$$-i\,\hat{\boldsymbol{k}} \cdot \frac{\boldsymbol{p}}{m} \, | \Phi_{0\alpha}^{(0)} \rangle + \hat{C}^+ \, | \Phi_{0\alpha}^{(1)} \rangle = -i\ \omega_{0\alpha}^{(1)} \, | \Phi_{0\alpha}^{(0)} \rangle$$

$$\to \qquad \hat{C}^+ \, | \Phi_{0\alpha}^{(1)} \rangle = i \left[\hat{\boldsymbol{k}} \cdot \frac{\boldsymbol{p}}{m} - \omega_{0\alpha}^{(1)} \right] | \Phi_{0\alpha}^{(0)} \rangle \tag{11.166}$$

$$\therefore \qquad | \Phi_{0\alpha}^{(1)} \rangle = i\,\frac{\hat{Q}_0}{\hat{C}^+} \left[\hat{\boldsymbol{k}} \cdot \frac{\boldsymbol{p}}{m} - \omega_{0\alpha}^{(1)} \right] | \Phi_{0\alpha}^{(0)} \rangle$$
$$= i\,\frac{\hat{Q}_0}{\hat{C}^+}\, \hat{\boldsymbol{k}} \cdot \frac{\boldsymbol{p}}{m} \, | \Phi_{0\alpha}^{(0)} \rangle \tag{11.166c}$$

where $\hat{Q}_0 = 1 - \hat{P}_0$ is the exclusion operator that excludes the subspace spanned by $\{ \ | \phi_{0\alpha} \rangle \}$.

As in §11.E.2,, we assume the orthogonality conditions [see (11.119e)]

$$\langle \Phi_n^{(i)} | \Phi_n^{(j)} \rangle = 0 \qquad \text{if} \qquad i \neq j$$

(11.166d)

which implies

$$\langle \Phi_{0\alpha}^{(i)} | \Phi_{0\alpha'}^{(j)} \rangle = 0 \qquad \to \qquad \langle \phi_{0\alpha} | \Phi_{0\alpha'}^{(1)} \rangle = 0 \qquad \forall\ \alpha,\ \alpha'$$

(11.166e)

Now, $\langle\ \phi_{0\alpha''}\ |$ (11.166) $\rangle$ gives

$$\langle \phi_{0\alpha''} | \hat{C}^+ | \Phi_{0\alpha}^{(1)} \rangle = \langle \hat{C}^+ \phi_{0\alpha''} | \Phi_{0\alpha}^{(1)} \rangle = 0 \qquad [\ \text{(11.166b) used.}\ ]$$
$$= i \left\langle \phi_{0\alpha''} \right| \left[\hat{\boldsymbol{k}} \cdot \frac{\boldsymbol{p}}{m} - \omega_{0\alpha}^{(1)} \right] | \Phi_{0\alpha}^{(0)} \rangle$$

$$\rightarrow \qquad \omega_{0\,\alpha}^{(1)} \left\langle \phi_{0\,\alpha''} \,\middle|\, \Phi_{0\,\alpha}^{(0)} \right\rangle = \left\langle \phi_{0\,\alpha''} \,\middle|\, \hat{\boldsymbol{k}} \cdot \frac{\boldsymbol{p}}{m} \,\middle|\, \Phi_{0\,\alpha}^{(0)} \right\rangle \tag{11.167}$$

Using the expansion (11.164) & the orthonormality conditions (11.157a), we get

$$\begin{aligned} \omega_{0\,\alpha}^{(1)} c_{\alpha\,\alpha''} &= \sum_{\alpha'} c_{\alpha\,\alpha'} \left\langle \phi_{0\,\alpha''} \,\middle|\, \hat{\boldsymbol{k}} \cdot \frac{\boldsymbol{p}}{m} \,\middle|\, \phi_{0\,\alpha'} \right\rangle \\ &= \sum_{\alpha'} c_{\alpha\,\alpha'} W_{\alpha''\,\alpha'} \end{aligned}$$

(11.168)

where

$$\begin{aligned} W_{\alpha''\,\alpha'} &= \left\langle \phi_{0\,\alpha''} \,\middle|\, \hat{\boldsymbol{k}} \cdot \frac{\boldsymbol{p}}{m} \,\middle|\, \phi_{0\,\alpha'} \right\rangle \\ &= \left(\frac{\beta}{2\,\pi\,m}\right)^{3/2} \int d\,\boldsymbol{p}\; e^{-\beta\,p^2/2\,m}\, \phi_{0\,\alpha''}(\boldsymbol{p})\, \hat{\boldsymbol{k}} \cdot \frac{\boldsymbol{p}}{m}\, \phi_{0\,\alpha'}(\boldsymbol{p}) \end{aligned}$$

(11.169)

Note that the integral is non-vanishing only if the integrand is an even power in at least one factor p_j.

Let

$$\hat{\boldsymbol{k}} \cdot \boldsymbol{p} = p_j \qquad\qquad j = 1,\ 2,\ 3\ = x,\ y,\ z$$

then (11.169) becomes

$$W_{\alpha''\,\alpha'} = \left(\frac{\beta}{2\,\pi\,m}\right)^{3/2} \int d\,\boldsymbol{p}\; e^{-\beta\,p^2/2\,m}\, \phi_{0\,\alpha''}(\boldsymbol{p})\, \frac{p_j}{m}\, \phi_{0\,\alpha'}(\boldsymbol{p})$$

If the integral is to be non-vanishing, one of the $\phi_{0\,\alpha}$ must pair with p_j so that it has to be $\phi_{0,\,j+1}$. The other $\phi_{0\,\alpha}$ is then either ϕ_{01} or ϕ_{05}. The only non-vanishing terms are therefore

$$\begin{aligned} W_{1,\,j+1} &= W_{j+1,1} \\ &= \left(\frac{\beta}{2\,\pi\,m}\right)^{3/2} \int d\,\boldsymbol{p}\; e^{-\beta\,p^2/2\,m}\, \frac{p_j^2}{m} \sqrt{\frac{\beta}{m}} && \text{[see (11.158-61)]} \\ &= \frac{1}{\beta} \sqrt{\frac{\beta}{m}} && \text{[(11.157d) used.]} \end{aligned}$$

(11.170)

and

$$\begin{aligned} W_{5,\,j+1} &= W_{j+1,5} \\ &= \left(\frac{\beta}{2\,\pi\,m}\right)^{3/2} \int d\,\boldsymbol{p}\; e^{-\beta\,p^2/2\,m}\, \frac{p_j^2}{m} \sqrt{\frac{\beta}{m}} \sqrt{\frac{2}{3}} \left(-\frac{3}{2} + \frac{\beta}{2\,m} p^2\right) && \text{[(11.162) used.]} \\ &= \frac{1}{m} \sqrt{\frac{\beta}{m}} \sqrt{\frac{2}{3}} \left\{ -\frac{3}{2} \frac{m}{\beta} + \frac{\beta}{2\,m} \left[2 \left(\frac{m}{\beta}\right)^2 + 3 \left(\frac{m}{\beta}\right)^2 \right] \right\} && \text{[(11.157d \& f) used.]} \\ &= \sqrt{\frac{2}{3\,m\,\beta}} \end{aligned}$$

(11.171)

(11.168) can be written as

$$\sum_{\alpha'} \left(W_{\alpha''\alpha'} - \delta_{\alpha''\alpha'}\, \omega^{(1)}_{0\,\alpha} \right) c_{\alpha\alpha'} = 0$$

which corresponds to the matrix form

$$\left(\mathbb{W} - \omega^{(1)}_{0\,\alpha}\, \mathbb{I} \right) \mathbb{c}_\alpha = \mathbb{0}$$

where

$$\mathbb{c}_\alpha = \left(c_{\alpha 1},\ \ldots,\ c_{\alpha 5} \right)^T$$

Thus, $\omega^{(1)}_{0\,\alpha}$ are the eigenvalues of $\mathbb{W}$ satisfying the eigen-equation

$$\left(\mathbb{W} - \lambda\, \mathbb{I} \right) \mathbb{c} = \mathbb{0} \qquad \mathbb{c} = \left(c_1,\ \ldots,\ c_5 \right)^T$$

i.e., they are the roots of the secular equation

$$\det \left| \ \mathbb{W} - \lambda\, \mathbb{I} \ \right| = 0$$

(11.172)

Setting $j = 1$, we have

$$\hat{\boldsymbol{k}} \cdot \boldsymbol{p} = p_1 = p_x$$

we have

$$\mathbb{W} = \frac{1}{\sqrt{m\,\beta}} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \sqrt{2/3} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2/3} & 0 & 0 & 0 \end{pmatrix}$$

Using the *Mathematica* code in §Code, we get

$$\left\{ \omega^{(1)}_{01},\ \omega^{(1)}_{02},\ \omega^{(1)}_{03},\ \omega^{(1)}_{04},\ \omega^{(1)}_{05} \right\} = \sqrt{\frac{5}{3\,m\,\beta}} \left\{ -1,\ 1,\ 0,\ 0,\ 0 \right\} \qquad (11.174\text{-}5)$$

$$= c \left\{ -1,\ 1,\ 0,\ 0,\ 0 \right\}$$

where [see §(Speed of Sound)]

$$c = \sqrt{\left(\frac{\partial P}{\partial \rho} \right)_s} = \sqrt{\frac{5}{3\,m\,\beta}} = \text{speed of sound of ideal gas}$$

Thus, the 1st order perturbation lifts the degeneracy of only two of the zero-frequency modes. Higher order terms in the perturbation are needed to lift the degeneracy completely.

The eigenvectors for the eigenvalues in (11.174-5) are [see §Code]

$$\left\{ \begin{pmatrix} \sqrt{3/2} \\ -\sqrt{5/2} \\ 0 \\ 0 \\ 1 \end{pmatrix},\ \begin{pmatrix} \sqrt{3/2} \\ \sqrt{5/2} \\ 0 \\ 0 \\ 1 \end{pmatrix},\ \begin{pmatrix} -\sqrt{3/2} \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},\ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

which, after normalization, becomes

$$\left\{ \begin{pmatrix} \sqrt{3/10} \\ -\sqrt{1/2} \\ 0 \\ 0 \\ \sqrt{1/5} \end{pmatrix}, \begin{pmatrix} \sqrt{3/10} \\ \sqrt{1/2} \\ 0 \\ 0 \\ \sqrt{1/5} \end{pmatrix}, \begin{pmatrix} -\sqrt{2/5} \\ 0 \\ 0 \\ 0 \\ \sqrt{3/5} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

(11.175a)

In terms of the basis $\{ \mid \phi_{0\alpha} \rangle \}$, they correspond to

$$\omega_{01}^{(1)} = -c : \qquad \left| \Phi_{01}^{(0)} \right\rangle = \frac{1}{\sqrt{2}} \left(\sqrt{\frac{3}{5}} \left| \phi_{01} \right\rangle - \left| \phi_{02} \right\rangle + \sqrt{\frac{2}{5}} \left| \phi_{05} \right\rangle \right)$$

(11.177)

$$\omega_{02}^{(1)} = c: \qquad \left| \Phi_{02}^{(0)} \right\rangle = \frac{1}{\sqrt{2}} \left(\sqrt{\frac{3}{5}} \left| \phi_{01} \right\rangle + \left| \phi_{02} \right\rangle + \sqrt{\frac{2}{5}} \left| \phi_{05} \right\rangle \right)$$

(11.176)

$$\omega_{03}^{(1)} = 0: \qquad \left| \Phi_{03}^{(0)} \right\rangle = -\sqrt{\frac{2}{5}} \left| \phi_{01} \right\rangle + \sqrt{\frac{3}{5}} \left| \phi_{05} \right\rangle$$

(11.180)

$$\omega_{04}^{(1)} = 0: \qquad \left| \Phi_{04}^{(0)} \right\rangle = \left| \phi_{04} \right\rangle$$

(11.179)

$$\omega_{05}^{(1)} = 0: \qquad \left| \Phi_{05}^{(0)} \right\rangle = \left| \phi_{03} \right\rangle$$

(11.178)

Note that our results are the same as Reichl's except for the immaterial ordering of the eigenvalues.

Adapting the 2nd order correction [see (11.122) of §11.E.2] to the present situation, we have

$$\omega_{0\alpha}^{(2)} = \left\langle \Phi_{0\alpha}^{(0)} \right| \frac{p_x}{m} \frac{i \hat{Q}_0}{\hat{C}^+} \frac{p_x}{m} \left| \Phi_{0\alpha}^{(0)} \right\rangle$$

(11.181a)

where $\hat{Q}_0$ excludes the subspace spanned by $\{ \mid \phi_{0\alpha} \rangle \; ; \; \alpha = 1, \; ..., 5 \}$ or $\left\{ \left| \Phi_{0\alpha}^{(0)} \right\rangle \; ; \; \alpha = 1, \; ..., 5 \right\}$. As in §11.E.2, (11.181a) can be expressed as

$$\omega_{0\alpha}^{(2)} = \left\langle \Phi_{0\alpha}^{(0)} \right| \left(\frac{p_x}{m} - \omega_{0\alpha}^{(1)} \right) \frac{i}{\hat{C}^+} \left(\frac{p_x}{m} - \omega_{0\alpha}^{(1)} \right) \left| \Phi_{0\alpha}^{(0)} \right\rangle \qquad (11.181b)$$

where, since $\mathbb{W}$ is diagonal with respect to the basis $\left\{ \left| \Phi_{0\alpha}^{(0)} \right\rangle \right\}$ [see (11.169)],

$$\omega_{0\alpha}^{(1)} = \left\langle \Phi_{0\alpha}^{(0)} \right| \frac{p_x}{m} \left| \Phi_{0\alpha}^{(0)} \right\rangle$$

Thus, to $O(k^2)$, we have

$$\omega_{01}(\boldsymbol{k}) = -c\,k + i\,k^2 \left\langle \Phi_{01}^{(0)} \right| \frac{p_x}{m} \frac{\hat{Q}_0}{\hat{C}^+} \frac{p_x}{m} \left| \Phi_{01}^{(0)} \right\rangle$$

$$= -c\,k + i\,k^2 \left\langle \Phi_{01}^{(0)} \middle| \left(\frac{p_x}{m} + c\right) \frac{1}{\hat{C}^+} \left(\frac{p_x}{m} + c\right) \middle| \Phi_{01}^{(0)} \right\rangle \qquad (11.182)$$

$$\omega_{02}(\boldsymbol{k}) = c\,k + i\,k^2 \left\langle \Phi_{02}^{(0)} \middle| \frac{p_x}{m} \frac{\hat{Q}_0}{\hat{C}^+} \frac{p_x}{m} \middle| \Phi_{02}^{(0)} \right\rangle$$

$$= c\,k + i\,k^2 \left\langle \Phi_{02}^{(0)} \middle| \left(\frac{p_x}{m} - c\right) \frac{1}{\hat{C}^+} \left(\frac{p_x}{m} - c\right) \middle| \Phi_{02}^{(0)} \right\rangle$$

(11.181)

$$\omega_{03}(\boldsymbol{k}) = i\,k^2 \left\langle \Phi_{03}^{(0)} \middle| \frac{p_x}{m} \frac{\hat{Q}_0}{\hat{C}^+} \frac{p_x}{m} \middle| \Phi_{03}^{(0)} \right\rangle$$

$$= i\,k^2 \left\langle \Phi_{03}^{(0)} \middle| \frac{p_x}{m} \frac{1}{\hat{C}^+} \frac{p_x}{m} \middle| \Phi_{03}^{(0)} \right\rangle$$

(11.185)

$$\omega_{04}(\boldsymbol{k}) = i\,k^2 \left\langle \Phi_{04}^{(0)} \middle| \frac{p_x}{m} \frac{\hat{Q}_0}{\hat{C}^+} \frac{p_x}{m} \middle| \Phi_{04}^{(0)} \right\rangle$$

$$= i\,k^2 \left\langle \Phi_{04}^{(0)} \middle| \frac{p_x}{m} \frac{1}{\hat{C}^+} \frac{p_x}{m} \middle| \Phi_{04}^{(0)} \right\rangle$$

(11.183)

$$\omega_{05}(\boldsymbol{k}) = i\,k^2 \left\langle \Phi_{05}^{(0)} \middle| \frac{p_x}{m} \frac{\hat{Q}_0}{\hat{C}^+} \frac{p_x}{m} \middle| \Phi_{05}^{(0)} \right\rangle$$

$$= i\,k^2 \left\langle \Phi_{05}^{(0)} \middle| \frac{p_x}{m} \frac{1}{\hat{C}^+} \frac{p_x}{m} \middle| \Phi_{05}^{(0)} \right\rangle$$

(11.184)

Owing to the linear k dispersion, ω_{01} & ω_{02} are sound modes. The dependence of its eigenvector on $|\,\phi_{01}\rangle$ & $|\,\phi_{05}\rangle$ [see (11.180) & (11.158-62)] makes ω_{03} the heat mode. Similarly, the dependence of their eigenvectors on $|\,\phi_{04}\rangle$ & $|\,\phi_{03}\rangle$ [see (11.178-9) & (11.158-62)] makes ω_{04} & ω_{05} the shear modes.

Speed of Sound

From Ex.2.8, answer (d), of §2.G.2, we have

$$\left(\frac{\partial V}{\partial P}\right)_s = -\frac{3\,V}{5\,P}$$

$$\rho = \frac{M}{V} \qquad \rightarrow \qquad -\frac{M}{\rho^2}\left(\frac{\partial \rho}{\partial P}\right)_s = -\frac{3\,M}{5\,P\,\rho}$$

$$\therefore \qquad \left(\frac{\partial P}{\partial \rho}\right)_s = \frac{5\,P}{3\,\rho}$$

$$= \frac{5\,N\,k_B\,T\,/\,V}{3\,M\,/\,V} = \frac{5\,k_B\,T}{3\,m} = \frac{5}{3\,m\,\beta}$$

Code

$$\mathbb{W} = \frac{1}{\sqrt{m\,\beta}} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \sqrt{2/3} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2/3} & 0 & 0 & 0 \end{pmatrix};$$

```
{λ, ev} = Eigensystem[W]
```

$$\left\{\left\{-\frac{\sqrt{\frac{5}{3}}}{\sqrt{m\,\beta}}, \frac{\sqrt{\frac{5}{3}}}{\sqrt{m\,\beta}}, 0, 0, 0\right\}, \left\{\left\{\sqrt{\frac{3}{2}}, -\sqrt{\frac{5}{2}}, 0, 0, 1\right\}, \left\{\sqrt{\frac{3}{2}}, \sqrt{\frac{5}{2}}, 0, 0, 1\right\}, \left\{-\sqrt{\frac{2}{3}}, 0, 0, 0, 1\right\}, \{0, 0, 0, 1, 0\}, \{0, 0, 1, 0, 0\}\right\}\right\}$$

```
(* Normalized to 1 *)
```

$$\frac{\#}{\sqrt{\#.\#}} \,\&\, /@\, ev$$

$$\left\{\left\{\sqrt{\frac{3}{10}}, -\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{5}}\right\}, \left\{\sqrt{\frac{3}{10}}, \frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{5}}\right\}, \left\{-\sqrt{\frac{2}{5}}, 0, 0, 0, \sqrt{\frac{3}{5}}\right\}, \{0, 0, 0, 1, 0\}, \{0, 0, 1, 0, 0\}\right\}$$